Linear Algebra (Review)

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Vectors

- $k, M, N$ are scalars
- A order-1 array $c$ is a column vector. Thus with two dimensions,
  \[ c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \]
  (more precisely, it is a representation of a vector in a specific coordinate system)
- $c_i$ is the $i$-th component of $c$
• $c^T = (c_1, c_2)$ is a row vector, the transposed of $c$
Matrices

• An order-2 array $A$ is a matrix, e.g.,

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

• We also write: $A = (a_{1,1}, a_{1,2}, a_{1,3}; a_{2,1}, a_{2,2}, a_{2,3})$; thus the semicolon “;” indicates row separations

• The colon “;” is sometimes used to select rows or columns; examples

$$A_{:,1} = \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} \quad A_{1,:} = (a_{1,1}, a_{1,2}, a_{1,3})$$
• If $A$ is an $N \times M$-dimensional matrix,

  – then the transposed $A^T$ is an $M \times N$-dimensional matrix

  – the columns (rows) of $A$ are the rows (columns) of $A^T$ and vice versa

\[
A^T = \begin{bmatrix}
  a_{1,1} & a_{2,1} \\
  a_{1,2} & a_{2,2} \\
  a_{1,3} & a_{2,3}
\end{bmatrix}
\]
Addition of Two Vectors

- Let $c = a + d$
- Then $c_j = a_j + d_j$
Multiplication of a Vector with a Scalar

- $c = ka$ is a vector with $c_j = ka_j$
- $C = kA$ is a matrix of the dimensionality of $A$, with $c_{i,j} = ka_{i,j}$
Scalar Product of Two Vectors

- The **scalar product** (also called dot product) is defined as

\[ a \cdot c = a^T c = \sum_{j=1}^{M} a_j c_j \]

and is a scalar.

- Special case:

\[ a^T a = \sum_{j=1}^{M} a_j^2 = ||a||^2 \]
Matrix-Vector Product

- A matrix consists of many row vectors. So a product of a matrix with a column vector consists of many scalar products of vectors

- If $A$ is an $N \times M$-dimensional matrix and $c$ is an $M$-dimensional column vector

- Then $d = Ac$ is an $N$-dimensional column vector $d$ with

$$d_i = \sum_{j=1}^{M} a_{i,j} c_j$$
Matrix-Matrix Product

• A matrix also consists of many column vectors. So a product of matrix with a matrix consists of many matrix-vector products

• If $A$ is an $N \times M$-dimensional matrix and $C$ an $M \times K$-dimensional matrix

• Then $D = AC$ is an $N \times K$-dimensional matrix with

$$d_{i,k} = \sum_{j=1}^{M} a_{i,j}c_{j,k}$$
Multiplication of a Row-Vector with a Matrix

- **Multiplication of a row vector with a matrix is a row vector.** If $A$ is a $N \times M$-dimensional matrix and $d$ a $N$-dimensional vector and if

$$c^T = d^T A$$

Then $c$ is a $M$-dimensional vector with $c_j = \sum_{i=1}^{N} d_i a_{i,j}$
Outer Product

- Special case: **Multiplication of a column vector with a row vector is a matrix.**
  Let $\mathbf{d}$ be a $N$-dimensional vector and $\mathbf{c}$ be a $M$-dimensional vector, then
  
  $$
  \mathbf{A} = \mathbf{d}\mathbf{c}^T
  $$

  is an $N \times M$ matrix with $a_{i,j} = d_i c_j$

Example:

$$
\begin{bmatrix}
    d_1 c_1 & d_1 c_2 & d_1 c_3 \\
    d_2 c_1 & d_2 c_2 & d_2 c_3 
\end{bmatrix} = 
\begin{bmatrix}
    d_1 \\
    d_2 
\end{bmatrix} 
\begin{bmatrix}
    c_1 & c_2 & c_3 
\end{bmatrix}
$$
Matrix Transposed

- The transposed $A^T$ changes rows and columns
- We have

$$\left( A^T \right)^T = A$$

$$(AC)^T = C^T A^T$$
Unit Matrix

\[
I = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & 1 \end{pmatrix}
\]
Diagonal Matrix

- $N \times N$ diagonal matrix:

\[
A = \begin{pmatrix}
a_{1,1} & 0 & \cdots & 0 \\
0 & a_{2,2} & \cdots & 0 \\
& & \ddots & \vdots \\
0 & \cdots & 0 & a_{N,N}
\end{pmatrix}
\]
Matrix Inverse

• Let $A$ be an $N \times N$ square matrix

• If there is a unique inverse matrix $A^{-1}$, then we have

$$A^{-1}A = I \quad AA^{-1} = I$$

• If the corresponding inverse exist, $(AC)^{-1} = C^{-1}A^{-1}$

• and $A^{-1T} = A^{T^{-1}}$
Orthogonal Matrices

- **Orthogonal Matrix** (more precisely: orthonormal matrix): $R$ is a (quadratic) orthogonal matrix, if all columns are orthonormal. It follows (non-trivially) that all rows are orthonormal as well and

\[ R^T R = I \quad RR^T = I \quad R^{-1} = R^T \]  (1)