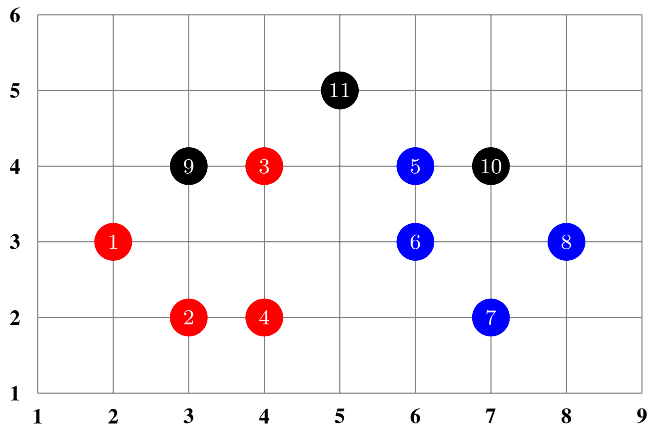
Ludwig-Maximilians-Universität München
Institut für Informatik
Prof. Dr. Thomas Seidl
Max Berrendorf, Julian Busch

Knowledge Discovery and Data Mining I
WS 2018/19

Exercise 11: SVM, Kernel Trick, Linear Separability

Exercise 11-1 Support Vector Machines



Consider the following training data:

$$x_1 = (2,3), x_2 = (3,2), x_3 = (4,4), x_4 = (4,2)$$

$$x_5 = (6,4), x_6 = (6,3), x_7 = (7,2), x_8 = (8,3)$$

Let $y_A = -1, y_B = +1$ be the class indicators for both classes

$$A = \{x_1, x_2, x_3, x_4\}, B = \{x_5, x_6, x_7, x_8\}.$$

(a) Just using the above-standing plot, specify which of the points should be identified as support vectors.

(b) Draw the maximum margin line which separates the classes (you don't have to do any computations here). Write down the normalized normal vector $\mathbf{w} \in \mathbb{R}^2$ of the separating line and the offset parameter $b \in \mathbb{R}$.

(c) Consider the decision rule: $H(x) = \langle \mathbf{w}, x \rangle + b$. Explain how this equation classifies points on either side of a line. Determine the class for the points $x_9 = (3,4)$, $x_{10} = (7,4)$ and $x_{11} = (5,5)$.

Exercise 11-2 Kernel Trick

Consider the polynomial kernel function

$$K : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto (x^T y + \gamma)^p, \text{ with } p = 2, \gamma = 1.$$

Furthermore let

$$\phi : \mathbb{R}^2 \to \mathbb{R}^6, x \mapsto (1, \sqrt{2}x_1, \sqrt{2}x_2, x_1^2, x_2^2, \sqrt{2}x_1x_2).$$

Show that $K(x, y) = \langle \phi(x), \phi(y) \rangle$.

Exercise 11-3 Mercer Kernels

As known from the lecture, a Mercer kernel $\kappa : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ needs to fulfil

(1) Symmetry, i.e., $\kappa(x, y) = \kappa(y, x)$

(2) Positive semi-definiteness, i.e. the kernel matrix $\kappa(X) := (\kappa(x_i, x_j))_{ij} \in \mathbb{R}^n$ is positive semi-definite for all $X = \{x_1, \ldots, x_n\} \subseteq \mathcal{X}$.

Show that the following functions are Mercer kernels for $x, y \in \mathcal{X} = \mathbb{R}^d$.

(a) $\kappa_1(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$.

(b) $\kappa_2(x, y) = x^T y$.

(c) $\kappa_3(x, y) = \alpha x^T y + \beta$ for $\alpha, \beta \in \mathbb{R}$ with $\alpha, \beta \geq 0$

Exercise 11-4 Linear Separability

In the following exercise, provide minimal subsets $\{x_1, \ldots, x_m\} = X \subseteq \mathcal{X} = \mathbb{R}^d$ together with class labels $y_1, \ldots, y_m \in \{-1, 1\}$ for the given dimensionality $d \in \mathbb{N}$ that are not linear separable. Prove both, the minimality (i.e. every $X' \subseteq \mathcal{X}$ with $|X'| < |X|$ is linearly separable), as well as the non-separability of X.

(a) $d = 1$

(b) $d = 2$