FAKULTÄT FÜR MATHEMATIK, INFORMATIK UND STATISTIK
INSTITUT FÜR INFORMATIK
LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
LEHRSTUHL FÜR DATENBANKSYSTEME UND DATA MINING

**Lecture Notes to**

Big Data Management and Analytics
Winter Term 2017/2018

# Community Detection



DBS

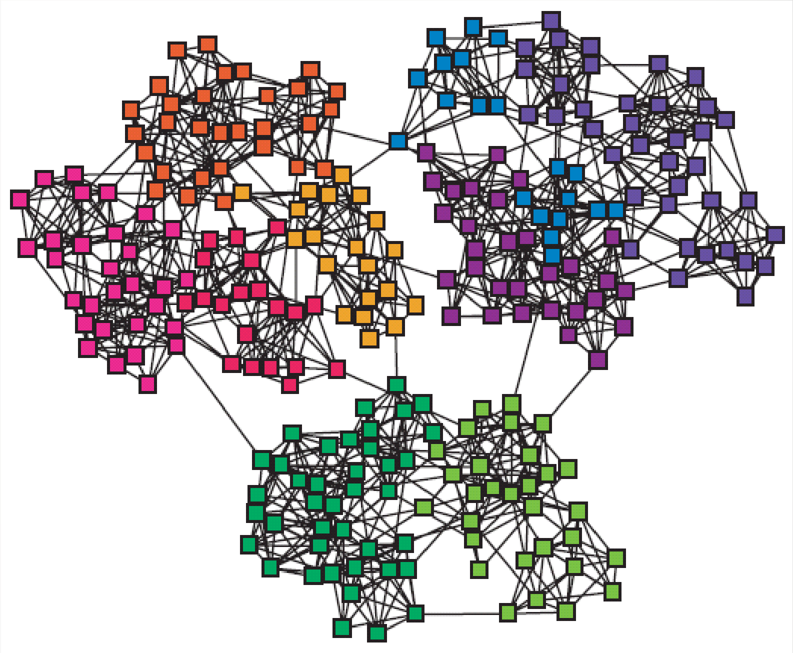
# Outline

- Community Detection
- Social networks
- Betweenness
  - Girvan-Newman Algorithm
- Modularity
- Graph Partitioning
  - Spectral Graph Partitioning
- Trawling

# Networks & Communities:

**Think of networks being organized into:**

- Modules
- Cluster
- Communities



**→ Goal: Find densely linked clusters**

# What is a Social Network?

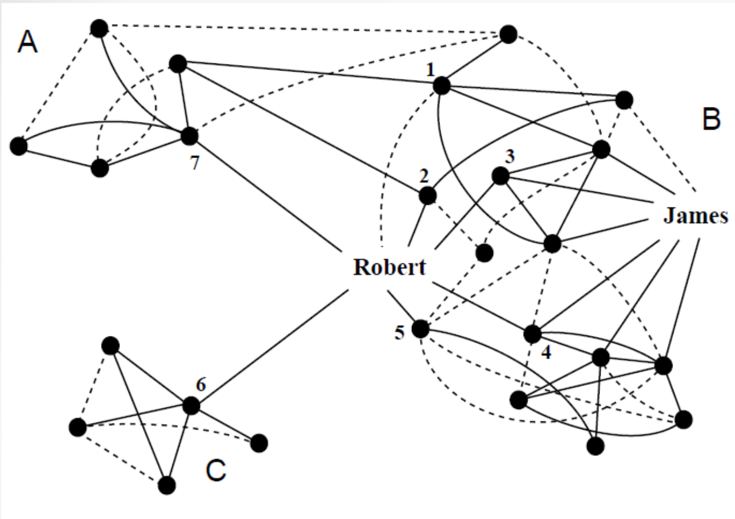
## Characteristics of a social network:

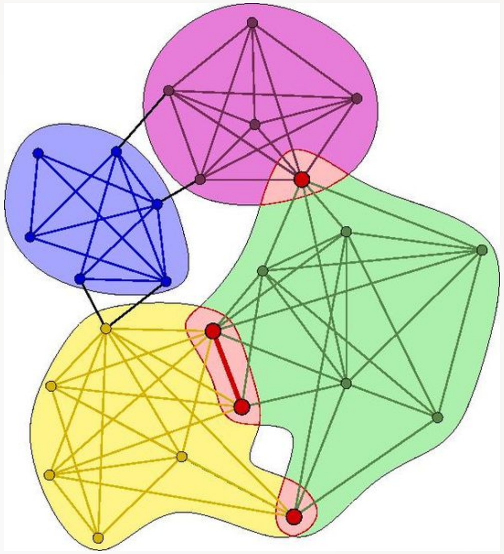
- **Collection of entities** participating in the network (entities might be individuals, phone numbers, email addresses , …)

- At least one **relationship between entities** of the network. (Facebook: ‘friend’). Relationship can be all-or-nothing or specified by a degree (e.g. fraction of the average day that two people communicate to each other)

- Assumption of **non-randomness** or **locality**, i.e. relationships tend to cluster. (e.g. A is related to B and C → higher probability that B is related to C)

# How to find communities?

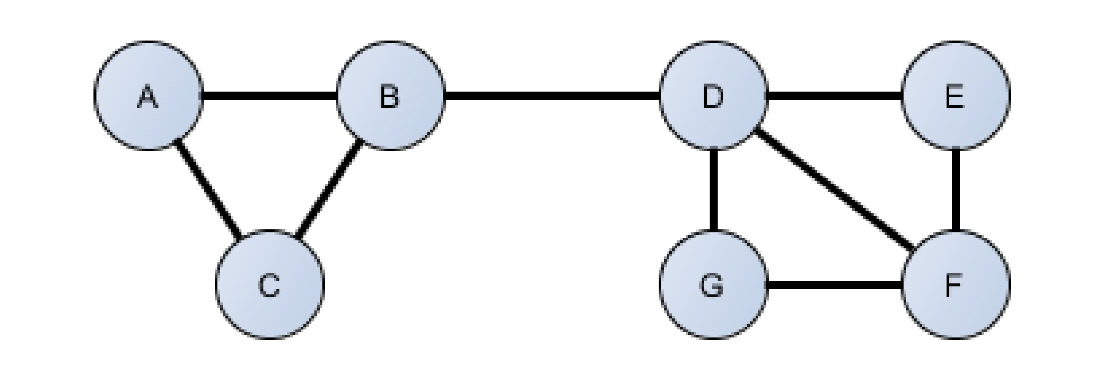


- Here we will work with undirected (unweighted networks)
- We need to resolve 2 questions:
  - How to compute betweenness?
  - How to select the number of clusters?

# Betweenness

**Definition:** The betweenness of an edge (a,b) is the number of pairs of nodes x and y such that (a,b) lies on the shortest-path between x and y.




**example:**
Edge (B,D) has the highest betweenness
(shortest path of A,B,C to any of D,E,F,G)
→ Betweenness of (B,D) aggregates to: 3 x 4 = 12

What is the betweenness of edge (D,F) ?

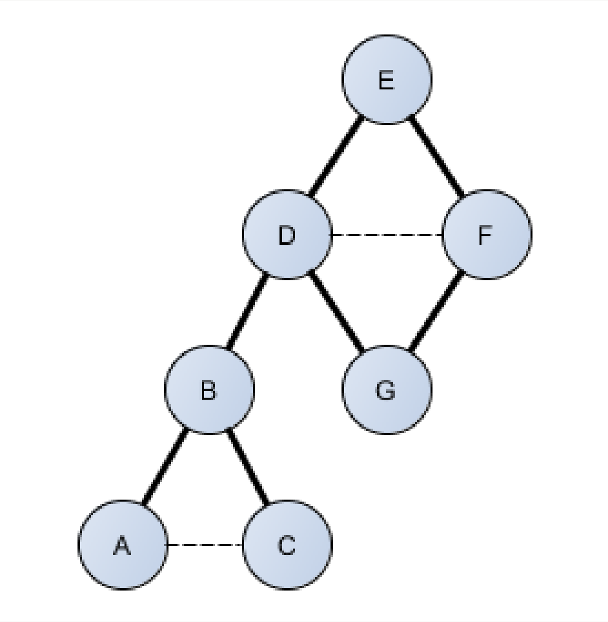
# Girvan-Newman Algorithm

**Goal: Computation of betweenness of edges**

**Step 1:** Perform a breadth-first search, starting at node X and construct a DAG (directed, acyclic graph)

**example:**

Start at node E




Level 1

Level 2

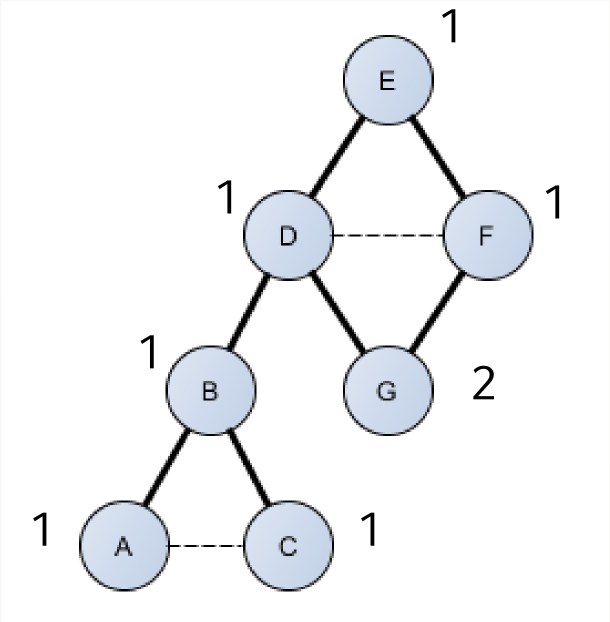
Level 3

# Girvan-Newman Algorithm

**Goal: Computation of betweenness of edges**

**Step 2:** label each node by the number of shortest paths that reach it from the root. Label of root = 1, each node is labeled by the sum of its parents.

**example:**




Level 1

Level 2

Level 3

# Girvan-Newman Algorithm

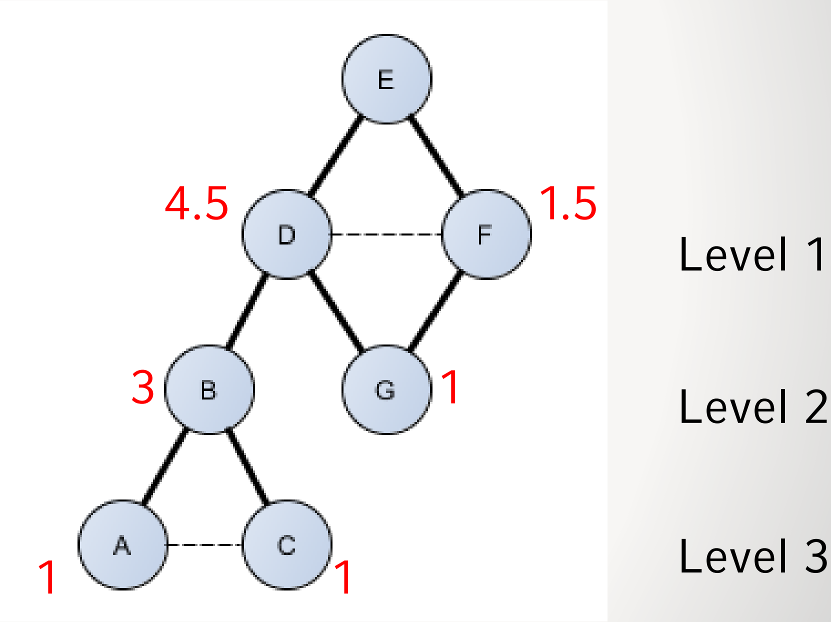
## Goal: Computation of betweenness of edges

**Step 3:** calculate for each edge e the sum over all nodes Y the fraction of shortest paths from the root X to Y:

**in detail:**

1. Each leaf gets a credit of 1.
2. Non-leaf nodes get a credit of 1 plus the sum of the credit of their children
3. A DAG edge $e$ entering node $Z$ from the level above is given a share of the credit of $Z$ proportional to the fraction of shortest paths from the root to $Z$.
   **Formally:** let $Y_1, \ldots, Y_k$ be the parent nodes of $Z$ with $p_i, 1 \leq i \leq k$, be the number of shortest path to $Y_i$. The credit for edge $(Y_i, Z)$ is given by:

$$Z * p_i \,/\, \sum_{j=1}^{k} p_j$$

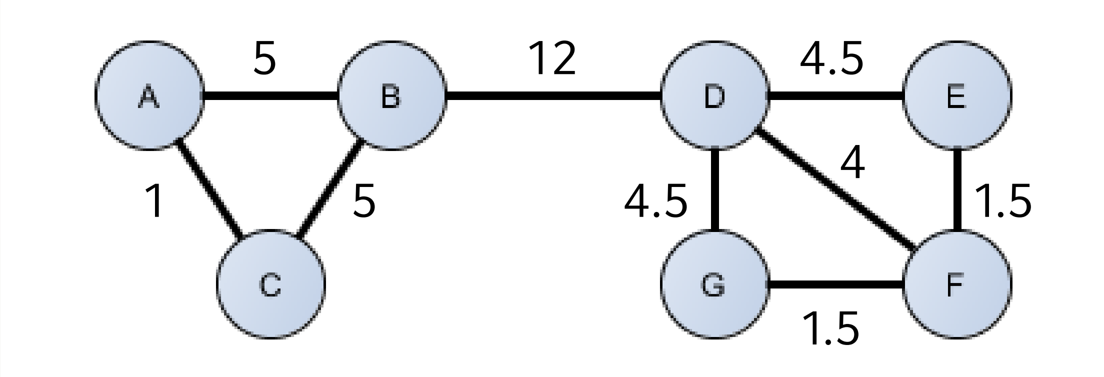


# Find Communities using Betweenness

**idea:** Clustering is performed by removing edges with the largest betweenness until separated communities remain.

**example:**
GN-Algorithm has been performed for every node and the credit of each edge has been calculated (by summing the credits up and dividing them by 2. Why?)

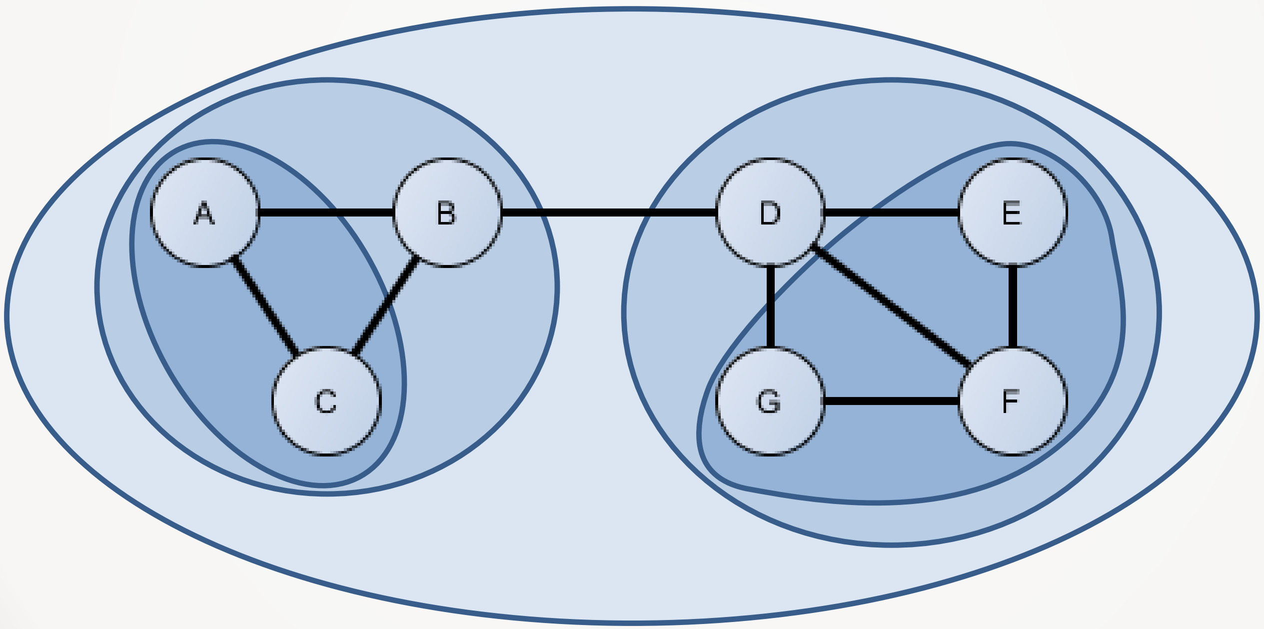
Remove edges, starting with highest betweenness:
- 1. Remove (B,D)
  → Communities {A,B,C} and {D,E,F,G}
- 2. Remove (A,B), (B,C), (D,G), (D,E), (D,F)
  → Communities {A,C} and {E,F,G}
  Node B and D are encapsulated as ‚traitors' of communities



# Find Communities using Betweenness

**Girvan-Newman Algorithm:**

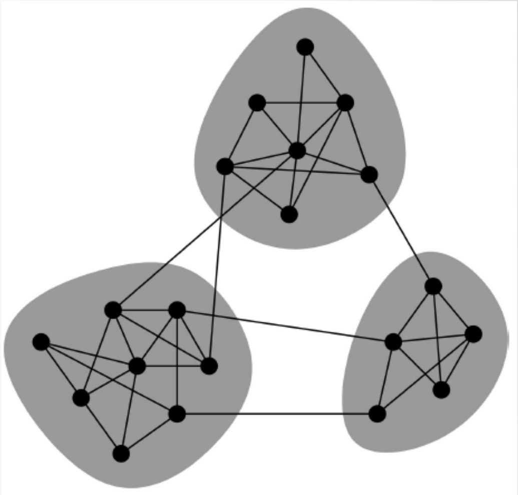
- connected components are communities
- gives a hierarchical decomposition of the network



# Analysis of Large Graphs

## Network Communities

✓ How to compute betweenness?
→ **How to select the number of clusters?**



**Communities:** sets of tightly connected nodes

**Modularity Q:**

- A measure of how well a network is partitioned into communities.
- Given a partitioning of the network into groups $s \in S$:

$$\boldsymbol{Q} \propto \sum_{s \in S} [(\#\boldsymbol{edges\ within\ group\ s}) - (\underbrace{\boldsymbol{expected\ \#edges\ within\ group\ s}}_{\text{defined by null model}})]$$

**defined by null model**

# Null Model: Configuration Model

Given a graph $G$ with $\boldsymbol{n}$ **nodes** and $\boldsymbol{m}$ **edges**, construct rewired network $G‘$:

- same degree distribution but random connections
- consider $G‘$ as a **multigraph**

→ The **expected number of edges between nodes $\boldsymbol{i}$ and $\boldsymbol{j}$** of degrees $k_i$ and $k_j$ is given by: $\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{m}} * \boldsymbol{k_i k_j}$

**Proof** that $G’$ contains the expected number of $m$ edges:

$$\frac{1}{2}\sum_{i\in N}\sum_{j\in N}\frac{k_i k_j}{2m} = \frac{1}{2}\frac{1}{2m}\sum_{i\in N} k_i \left(\sum_{j\in N} k_j\right) = \frac{1}{4m} * 2m * 2m = m$$

# Modularity

**Modularity of partitioning S of graph G:**

$$Q \propto \sum_{s \in S} [(\#edges\ within\ group\ s) - (expected\ \#edges\ within\ group\ s)]$$

$$Q(G,S) = \frac{1}{2m} \sum_{s \in S} \sum_{i \in s} \sum_{j \in s} \left(a_{ij} - \frac{k_i k_j}{2m}\right)$$

**Normalizing:** $-1 < Q < 1$

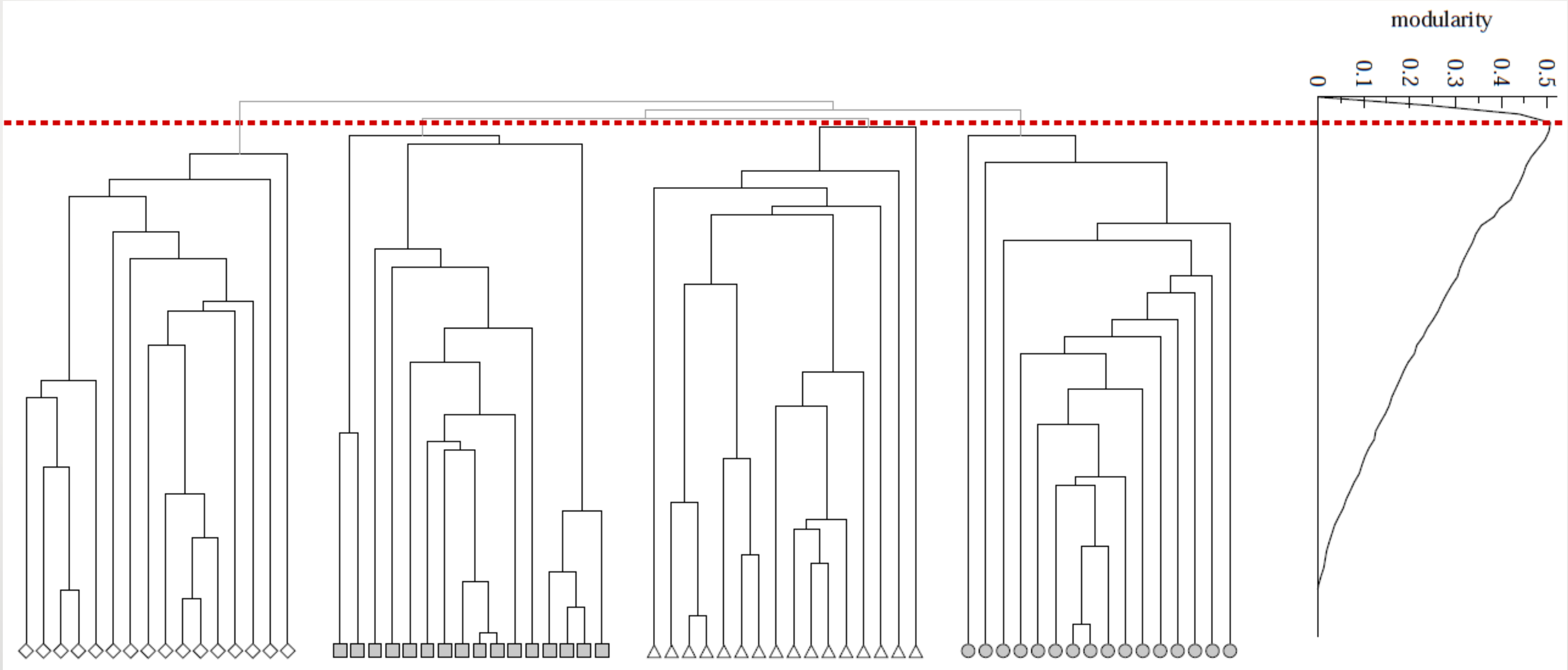
**Modularity values take range** $[-1, 1]$**:**

- positive if the number of edges within groups exceeds the expected number
- $0.3 - 0.7 < Q$ means significant community structure

# Analysis of Large Graphs

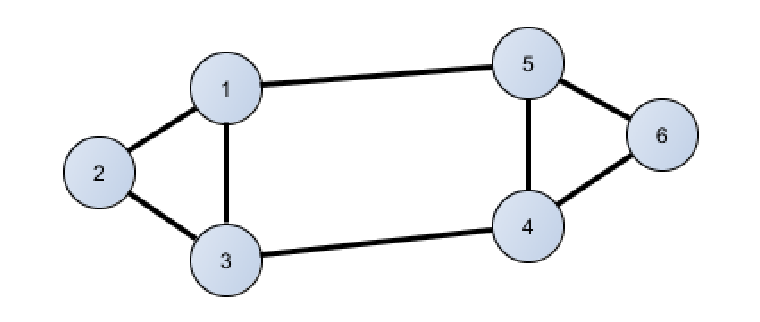
## Modularity

→ **Q is useful for selecting the number of clusters**



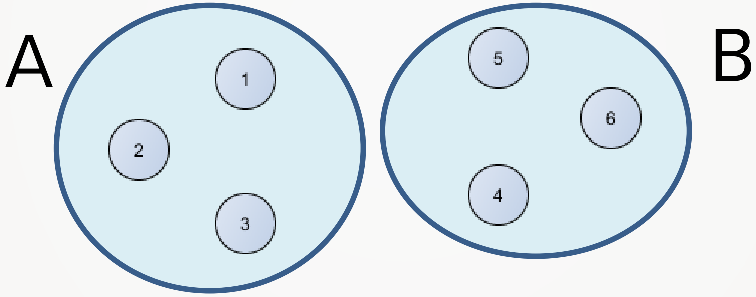
# Partitioning of Graphs

given an undirected Graph $G(V, E)$:




**bi-partitioning task:**

- Divide vertices into two **disjoint** groups $A, B$




**questions:**

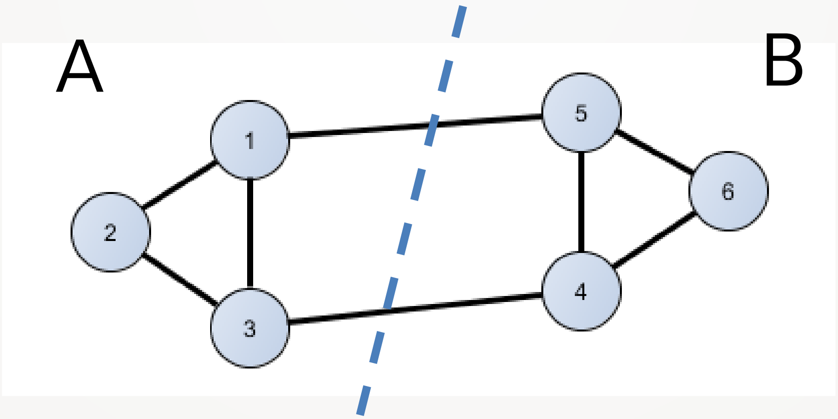
- How can we define 'good' partition of $G$?
- How can we efficiently identify such a partition?

# Partitioning of Graphs

What makes a good partition?

- Maximize the number of within-group connections
- Minimize the number of between-group connections

**example:**



# Partitioning of Graphs

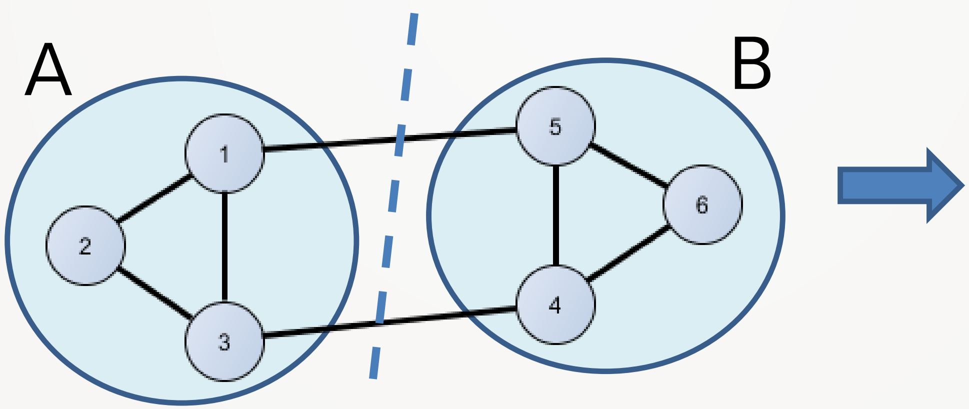
## Graph Cuts

Express partitioning objectives as a function of the 'edge cut' of the partition.

**Cut:** Set of edges with only one vertex in a group:

$$cut(A, B) = \sum_{i \in A, j \in B} w_{ij}$$

**example:**




$$\boldsymbol{cut(A, B) = 2}$$

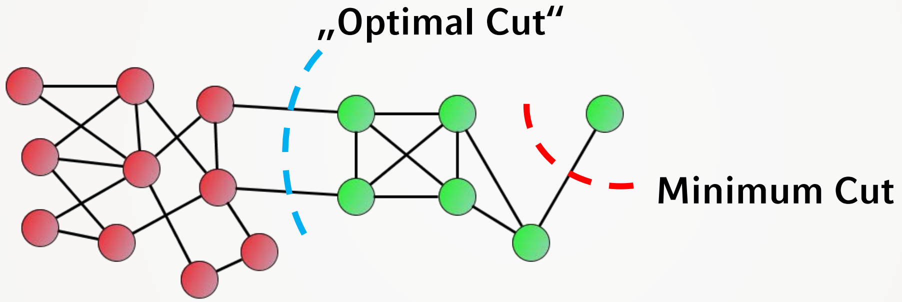
# Partitioning of Graphs

## Minimum-cut

minimize weight of connections between groups:

$$\underset{A,B}{\boldsymbol{arg\,min}}\ \boldsymbol{cut}(\boldsymbol{A},\boldsymbol{B})$$

**example**:




**problem:**

- only considers external cluster connections
- does not consider internal cluster connectivity

# Partitioning of Graphs - Graph Cuts

**Normalized-cut:** Connectivity between groups relative to the density of each group

$$\boldsymbol{ncut(A,B) = \frac{cut(A,B)}{vol(A)} + \frac{cut(A,B)}{vol(B)}}$$

$\boldsymbol{vol(X)}$ **:**total weight of edges with at least one endpoint in X: $vol(X) = \sum_{i \in A} k_i$

→ **Produces more balanced partitions**

**How to find a good partition efficiently?**
**Problem:** Computing optimal cuts is **NP-hard!**

# Spectral Graph Partitioning

**Given**

- **Adjacency matrix** of an undirected Graph G
  $a_{ij} = 1$ if $(i, j)$ exist in $G$, else 0
- Vector $x \in R^n$ with components $(x_1, \ldots, x_n)$
  Think of it as a label/value of each node of $G$

**What is the meaning of $\boldsymbol{A * x}$ ?**

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} * \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad y_i = \sum_{j=1}^{n} a_{ij} * x_j = \sum_{(i,j) \in E} x_j$$

**→ entry $\boldsymbol{y_i}$ is a sum of labels / values $\boldsymbol{x_j}$ of neighbors of $\boldsymbol{i}$**

# Spectral Graph Partitioning

## What is the meaning of $\boldsymbol{A} * \boldsymbol{x}$ ?

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} * \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} * \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\boldsymbol{A} * \boldsymbol{x} = \boldsymbol{\lambda} * \boldsymbol{x} \quad \boldsymbol{eigenvalue\ problem}$$

## Spectral Graph Theory:

- analyze the ‚spectrum' of matrix representing G
- **spectrum**: eigenvectors $x_i$ of a graph, ordered by the magnitude (strength) of their corresponding eigenvalues $\lambda_i$
- $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ with $\lambda_1 \leq \cdots \leq \lambda_n$

# Spectral Graph Partitioning

**Intuition**
Suppose all nodes in G have degree d and G is connected.

**What are some eigenvalues/vectors of G?**
eigenvalue problem: $A * x = \lambda * x$ → find $\lambda$ and $x$

- Let's try $x = (1, \dots, 1)$
- Then $A * x = (d, \dots, d) = \lambda * x$ → $\lambda = d$

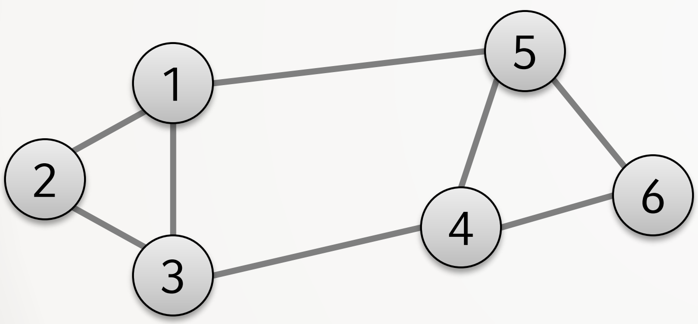
Remember:

$$y_i = \sum_{j=1}^{n} a_{ij} * x_j = \sum_{(i,j) \in E} x_j$$

# Spectral Graph Partitioning

**Adjacency matrix** A:

- $\boldsymbol{n}\ \boldsymbol{x}\ \boldsymbol{n}$ matrix
- $A = [a_{ij}], a_{ij} = 1$ if there is an edge between node $i$ and $j$

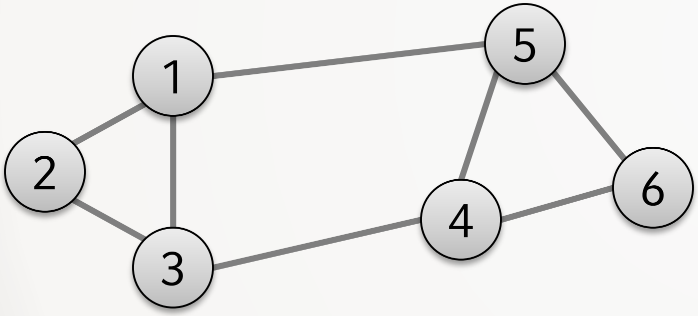



| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| 2 | 1 | 0 | 1 | 0 | 0 | 0 |
| 3 | 1 | 1 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 | 1 | 1 |
| 5 | 1 | 0 | 0 | 1 | 0 | 1 |
| 6 | 0 | 0 | 0 | 1 | 1 | 0 |

# Spectral Graph Partitioning

**Degree matrix** D:

- $\boldsymbol{n} \, \boldsymbol{x} \, \boldsymbol{n}$ diagonal matrix
- $D = [d_{ii}], d_{ii} =$ degree of node $i$

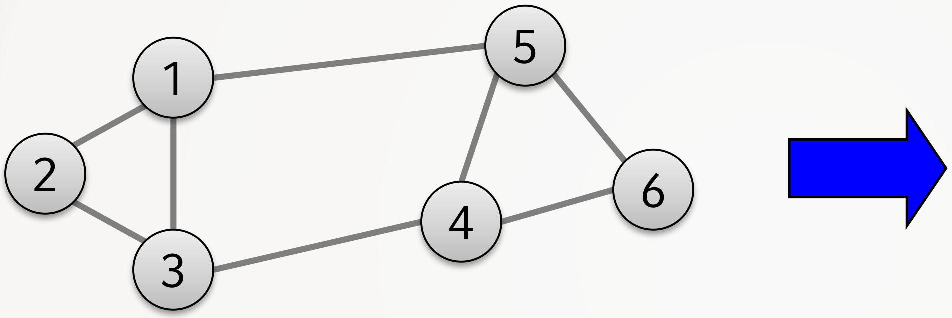



| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 2 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 3 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 3 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 3 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 2 |

# Spectral Graph Partitioning

**Laplacian Matrix** L:

- $\boldsymbol{n}\ \boldsymbol{x}\ \boldsymbol{n}$ symmetric matrix
- $L = D - A$




| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 3 | -1 | -1 | 0 | -1 | 0 |
| 2 | -1 | 2 | -1 | 0 | 0 | 0 |
| 3 | -1 | -1 | 3 | -1 | 0 | 0 |
| 4 | 0 | 0 | -1 | 3 | -1 | -1 |
| 5 | -1 | 0 | 0 | -1 | 3 | -1 |
| 6 | 0 | 0 | 0 | -1 | -1 | 2 |

- Trivial eigenpair?
  - $X = (1, \dots, 1)$ , then $L * x = 0$ and so $\lambda_1 = 0$

# Spectral Graph Partitioning

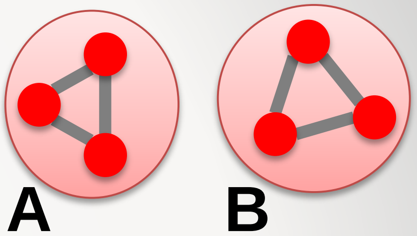
Now decompose the Laplacian instead of the adjacency matrix

**What are the eigenvalues/vectors of L?**
eigenvalue problem: $A * x = \lambda * x$ → find $\lambda$ and $x$

- Let's try $x = (1, \dots, 1)$
- Then $\mathrm{L} * x = (0, \dots, 0) = 0 * x$ → $\lambda = 0$
  (diagonal entry in row i: $L_{i,i} = -\sum_j X_{i,j}$)

=> The Laplacian of a connected graph has an eigenvalue 0 with a corresponding eigenvector (1,1,1,1,..,1)

# Spectral Graph Partitioning



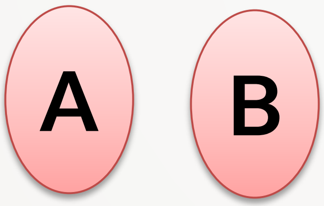

**Intuition**

**What if G is not connected?**

- G has 2 components, each d-regular

**What are some eigenvectors?**

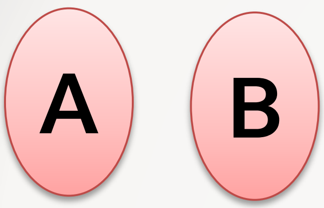
- $x =$ put all 1s on $A$ and 0s on $B$ or vice versa
  - $x' = (1, \dots, 1, 0, \dots, 0)$, then $A * x' = (d, \dots, d, 0, \dots, 0)$
  - $x'' = (0, \dots, 0, 1, \dots, 1)$, then $A * x'' = (0, \dots, 0, d, \dots, d)$
  - → in both cases the corresponding $\lambda = d$



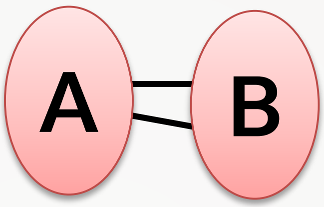

$\boldsymbol{\lambda_n = \lambda_{n-1}}$

# Spectral Graph Partitioning

## Intuition




$\boldsymbol{\lambda_n = \lambda_{n-1}}$




$\boldsymbol{\lambda_n - \lambda_{n-1} \approx 0}$

2nd largest eigenvalue $\lambda_{n-1}$ now has value very close to $\lambda_n$

- If the graph is connected (right example) then we already know that $x_n = (1, \dots, 1)$ is an eigenvector of L
- Since eigenvectors are orthogonal then the components of $x_{n-1}$ sum to 0
  - Why? → Because $\sum_i x_n[i] * x_{n-1}[i] = 0$ ($x_{n-1}$ must have negative components)
- **General Idea**: we can look at the eigenvector of the 2nd largest eigenvalue and declare nodes with positive label in $A$ and negative label in $B$

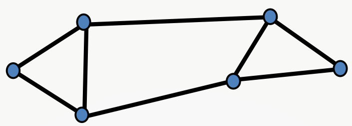
# Spectral Clustering Algorithms

**Three basic stages:**

1. **Pre-processing**
   - Construct a matrix representation of the graph
2. **Decomposition**
   - Compute eigenvalues and eigenvectors of the matrix
   - Map each point to a lower-dimensional representation on one or more eigenvectors
3. **Grouping**
   - Assign points to two or more clusters, based on the new representation

# Spectral Clustering Algorithms

1. **Pre-processing:**
   - Build Laplacian matrix L of the graph



| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 3 | -1 | -1 | 0 | -1 | 0 |
| 2 | -1 | 2 | -1 | 0 | 0 | 0 |
| 3 | -1 | -1 | 3 | -1 | 0 | 0 |
| 4 | 0 | 0 | -1 | 3 | -1 | -1 |
| 5 | -1 | 0 | 0 | -1 | 3 | -1 |
| 6 | 0 | 0 | 0 | -1 | -1 | 2 |

2. **Decomposition:**
   - Find eigenvalues $\lambda$ and eigenvectors $x$ of the matrix $L$
   - Map vertices to lower-dimensional representation

λ=

| |
|---|
| 0.0 |
| 1.0 |
| 3.0 |
| 3.0 |
| 4.0 |
| 5.0 |

X =

| | | | | | |
|---|---|---|---|---|---|
| 0.4 | 0.3 | -0.5 | -0.2 | -0.4 | -0.5 |
| 0.4 | 0.6 | 0.4 | -0.4 | 0.4 | 0.0 |
| 0.4 | 0.3 | 0.1 | 0.6 | -0.4 | 0.5 |
| 0.4 | -0.3 | 0.1 | 0.6 | 0.4 | -0.5 |
| 0.4 | -0.3 | -0.5 | -0.2 | 0.4 | 0.5 |
| 0.4 | -0.6 | 0.4 | -0.4 | -0.4 | 0.0 |

| | |
|---|---|
| **1** | 0.3 |
| **2** | 0.6 |
| **3** | 0.3 |
| **4** | -0.3 |
| **5** | -0.3 |
| **6** | -0.6 |

How do we now find the clusters?

# Spectral Clustering Algorithms

## 3. Grouping:

- Sort components of reduced 1-dimensional vector
- Identify clusters by splitting the sorted vector in two (threshold $\varepsilon$)
- By choosing $m$ vectors, there are max. $2^m$ clusters

→ **How to choose a splitting point, i.e threshold $\varepsilon$?**

- Naive approaches:
  - Split at $\varepsilon = 0$ or median value
- More expensive approaches:
  - Attempt to minimize normalized cut in 1-dimension
  - (sweep over ordering of nodes induces by the eigenvector)

| | |
|---|---|
| 1 | 0.3 |
| 2 | 0.6 |
| 3 | 0.3 |
| 4 | -0.3 |
| 5 | -0.3 |
| 6 | -0.6 |

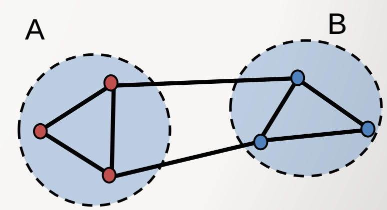
Split at $\varepsilon = 0$:

**Cluster A:** Positive points

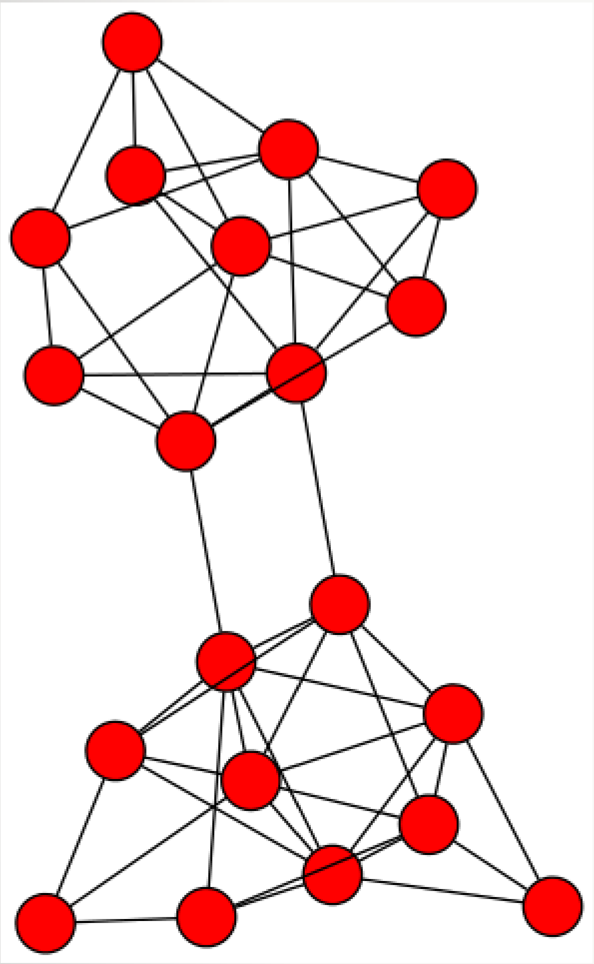
**Cluster B:** Negative points

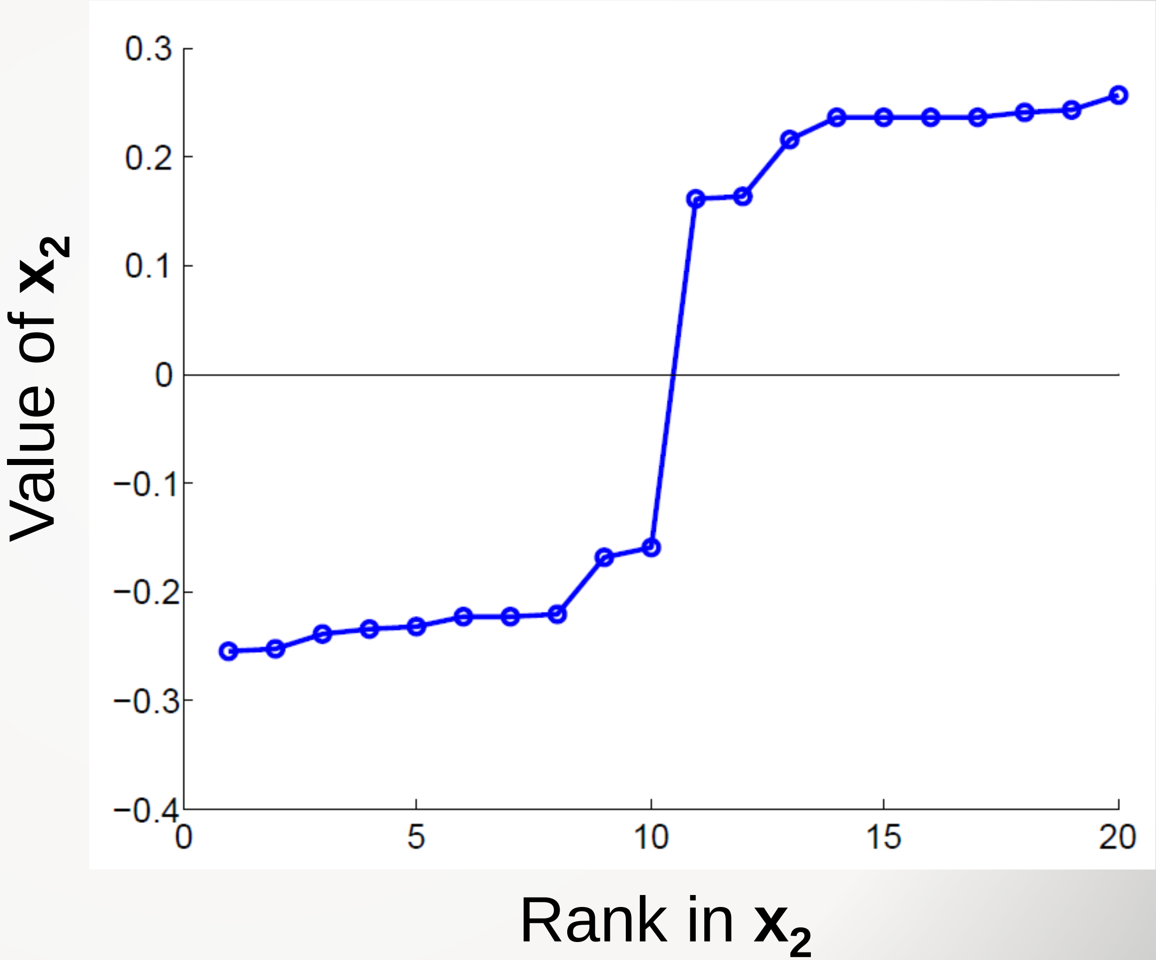
| | |
|---|---|
| 1 | **0.3** |
| 2 | **0.6** |
| 3 | **0.3** |

| | |
|---|---|
| 4 | **-0.3** |
| 5 | **-0.3** |
| 6 | **-0.6** |

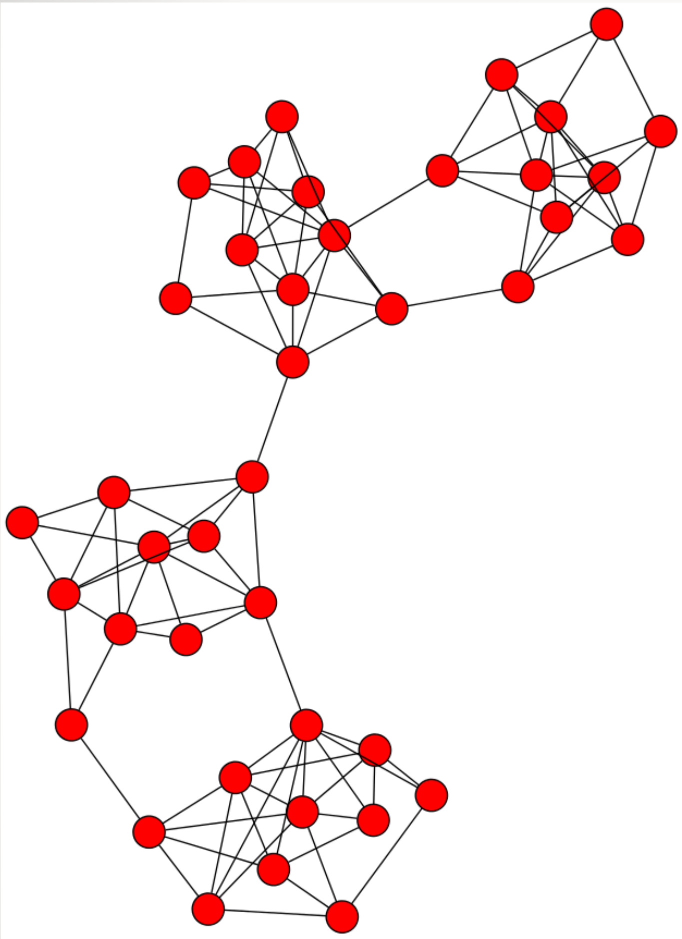


# Spectral Clustering Algorithms

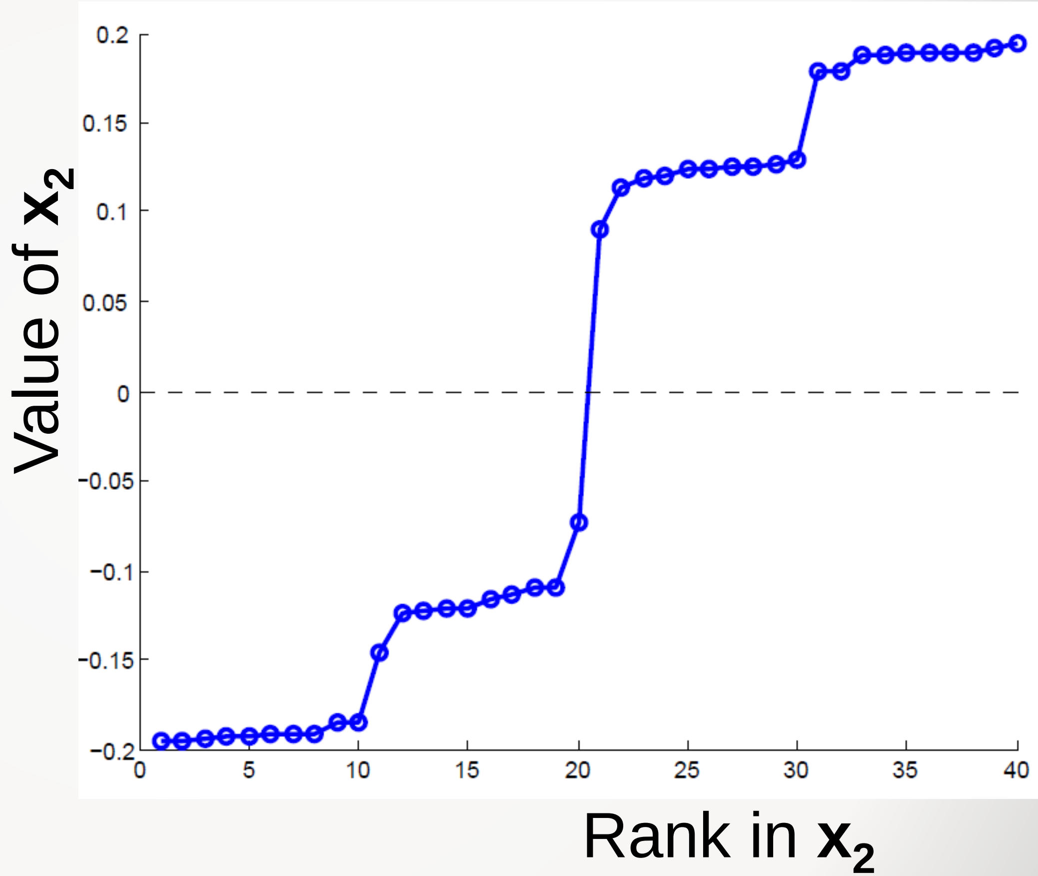




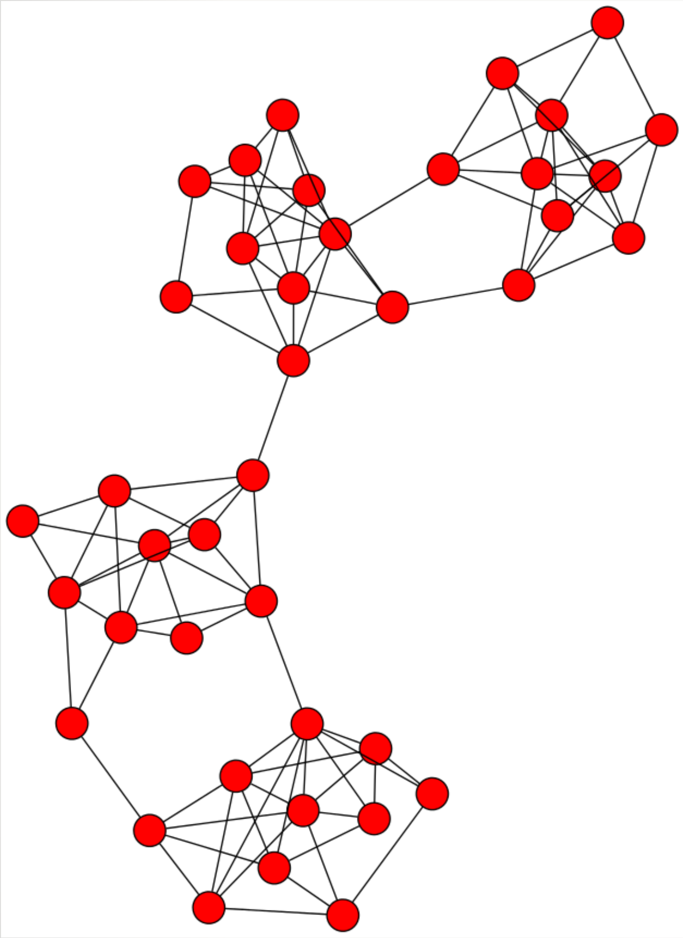
# Spectral Clustering Algorithms

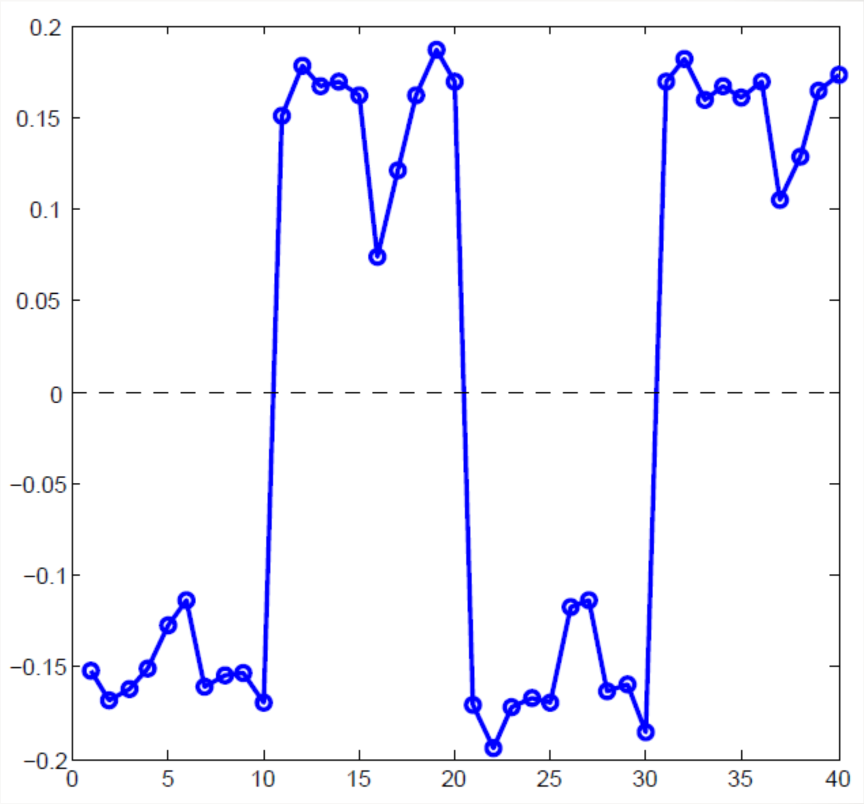


Components of $\mathbf{x_2}$



# Spectral Clustering Algorithms



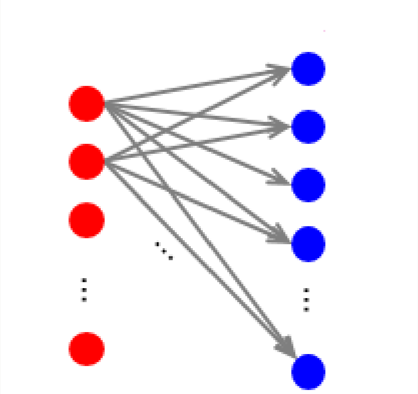



Components of $\mathbf{x}_3$

# Analysis of Large Graphs - Trawling

**Goal**: find small communities in huge graphs,
e.g. how to describe community/discussion in a Web

**example:**



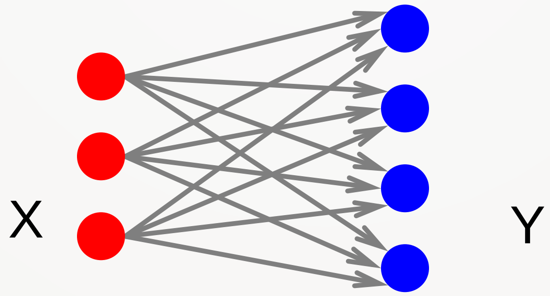
E.g. people talking about the same things or visited web pages

# Analysis of Large Graphs - Trawling

**Problem definition:**

Enumerate complete bipartite subgraphs $K_{s,t}$ :

- All vertices in $K_{s,t}$ can be partitioned in two sets. Each vertex in the first set of size ***s*** is linked to each vertex in second set of size ***t***
- Where $K_{s,t}$ : ***s*** nodes on the "left" where each links to the same ***t*** other nodes on the "right"



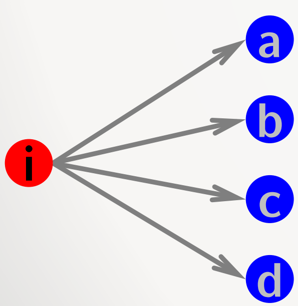

|X| = s = 3
|Y| = t = 4

# Analysis of Large Graphs - Trawling

## Frequent Itemset Analysis – Market Basket Analysis

- ***Market***: Universe $U$ of $n$ items
- ***Baskets***: subsets of $U$: $S_1, S_2, \ldots, S_m \subseteq U$
  - ($S_i$ is a set of items one person bought)
- ***Support:*** frequency threshold
- ***Goal:*** **Find all subsets $T$ s.t. $T \subseteq S_i$ of at least $f$ sets $S_i$**
  - (items in $T$ were bought together at least $f$ times)

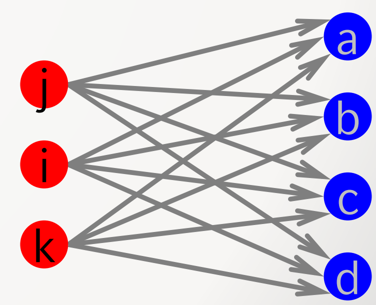
Frequent itemsets = complete bipartite graphs




**$S_i$={a,b,c,d}**

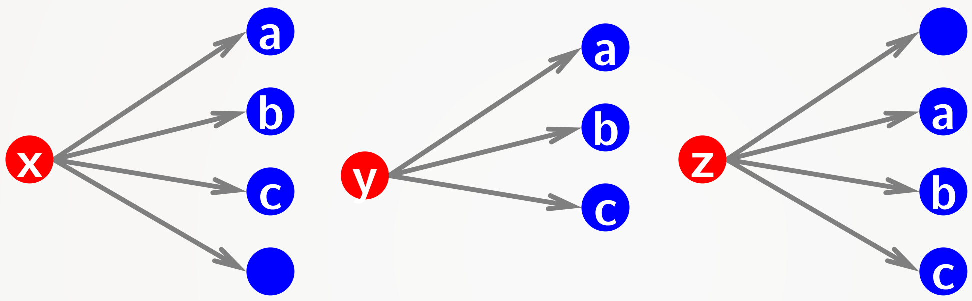
View each node $i$ as a set $S_i$ of nodes $i$ points to

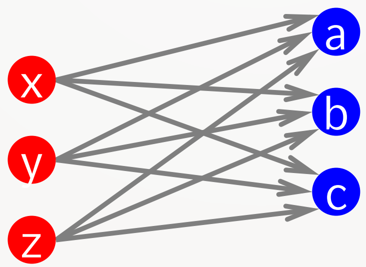
$K_{s,t}$ = a set $Y$ of size $t$ that occurs in $s$ sets $S_i$



# Analysis of Large Graphs - Trawling

E.g. Bipartite subgraph $K_{3,4}$ a **frequent itemset *Y={a,b,c}*** of supp ***s.*** So, there are ***s*** nodes that link to all of **{a,b,c}**:

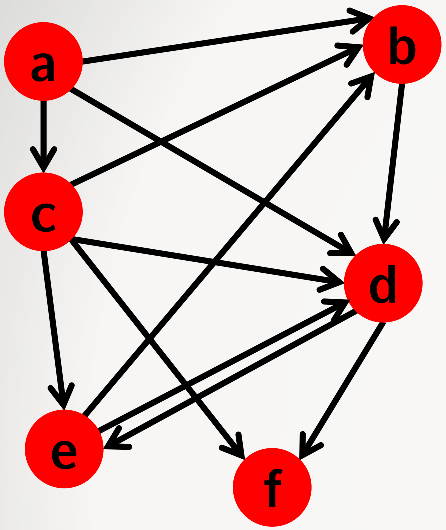



**We found $K_{s,t}$!**

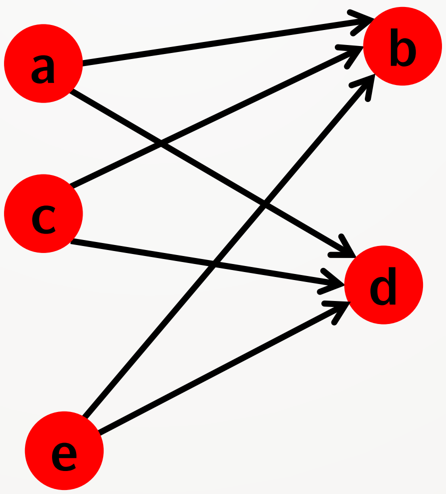
$K_{s,t}$ = a set ***Y*** of size ***t*** that occurs in *s* sets $S_i$

# Analysis of Large Graphs - Trawling




**Itemsets:**

a = {b,c,d}
b = {d}
c = {b,d,e,f}
d = {e,f}
e = {b,d}
f = {}




**Frequent itemsets support > 1**

**{b,d}**: support 3
**{e,f}**: support 2