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Managing Massive Multiplayer Online Games

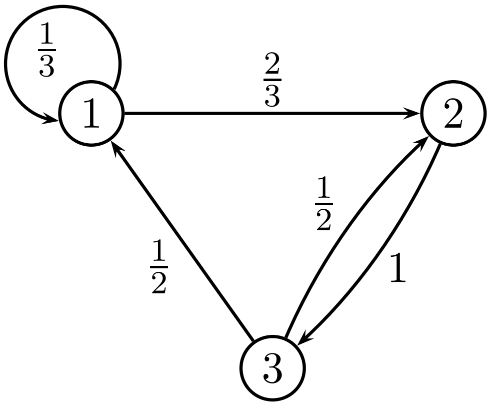
SS 2019

Exercise Sheet 8: Markov Chains

The assignments are due June 26, 2019

Assignment 8-1 *Markov Chains*

Given the Markov Chain M as depicted below. Nodes represent states, edges possible transitions and edge labels denote transition probabilities.



(a) Specify M in matrix notation. For that assume that the starting states are uniformly distributed and that sequences can only end after state 3 with a probability of 50%.

(b) What is the probability to observe the sequence $3-1-1-2-3$?

(c) What is the probability to observe the sequence $2-3-2-1-2$?

(a)

$$M = \begin{array}{c} \\ - \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} - & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{cccc} - & .33 & .33 & .33 \\ .0 & .33 & .66 & 0 \\ .0 & .0 & .0 & 1.0 \\ .5 & .25 & .25 & .0 \end{array} \right) \end{array}$$

(b)

$$\begin{aligned} P(3-1-1-2-3) &= P(3|-) \cdot P(1|3) \cdot P(1|1) \cdot P(2|1) \cdot P(3|2) \cdot P(-|3) \\ &= \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot 1.0 \cdot \frac{1}{2} = \frac{1}{216} \approx 0.00462963 \end{aligned}$$

(c)

$$
\begin{aligned}
P(2-3-2-1-2) &= P(2|-)\cdot P(3|2)\cdot P(2|3)\cdot P(1|2)\cdot P(2|1)\cdot P(-|2) \\
&= \frac{1}{3}\cdot 1.0\cdot\frac{1}{4}\cdot 0.0\cdot\frac{2}{3}\cdot 0.0 = 0.0
\end{aligned}
$$

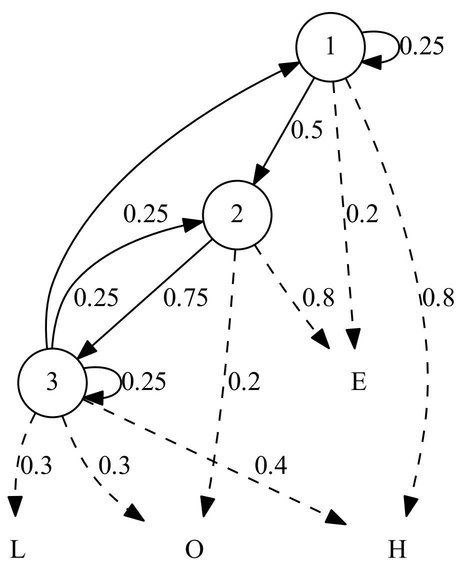
Assignment 8-2 *HMM: Calculation exercise*

Consider the Markov model given below.

(a) Specify the set of states A and the set of observations B. Deduce the transition matrix D and the output matrix F from the model. Assume that the starting probabilities are uniformly distributed and that the probabilities that sequences end in a state correspond to the values which are missing to the sum 1.

(b) Calculate the probability that the observation $O_1 = \{H, E, L, L, O\}$ is generated by the HMM.

(c) Which sequence $(s_1, s_2, \ldots, s_k)$ with $s_i \in A$ explains the observation $O_1 = \{H, E, L, L, O\}$ best?



(a) Specify the set of states A and the set of observations B. Deduce the transition matrix D and the output matrix F from the model. Assume that the starting probabilities are uniformly distributed and that the probabilities that sequences end in a state correspond to the values which are missing to the sum 1.

$$A = \{1, 2, 3\}, B = \{E, H, L, O\},$$

$$D = \begin{array}{c} \\ - \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} - & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} - & .33 & .33 & .33 \\ .25 & .25 & .5 & 0 \\ .25 & 0 & 0 & .75 \\ .25 & .25 & .25 & .25 \end{pmatrix} \end{array}, \quad F = \begin{array}{c} \times \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} E & H & L & O \end{array} \\ \begin{pmatrix} .2 & .8 & 0 & 0 \\ .8 & 0 & 0 & .2 \\ 0 & .4 & .3 & .3 \end{pmatrix} \end{array}$$

(b) Calculate the probability that the observation $O_1 = \{H, E, L, L, O\}$ is generated by the HMM. To determine the probability, we need to figure out the possible sequences of states that can produce the observation O_1. These are:

$1-2-3-3-2$,
$1-2-3-3-3$,
$3-2-3-3-2$, and
$3-2-3-3-3$.

Given the sequences of states, we can calculate the probability as follows.

$$\begin{aligned} P(HELLO) &= P(1-2-3-3-2) \cdot P(HELLO|1-2-3-3-2) \\ &+ P(1-2-3-3-3) \cdot P(HELLO|1-2-3-3-3) \\ &+ P(3-2-3-3-2) \cdot P(HELLO|3-2-3-3-2) \\ &+ P(3-2-3-3-3) \cdot P(HELLO|3-2-3-3-3) \\ &= 0.001933594 \cdot 0.01152 \\ &+ 0.001933594 \cdot 0.01728 \\ &+ 0.000966797 \cdot 0.00576 \\ &+ 0.000966797 \cdot 0.00864 = 0.000069609 \end{aligned}$$

Here, the single terms can be calculated as for instance

$$\begin{aligned} P(1-2-3-3-2) &= P(1|-) \cdot P(2|1) \cdot P(3|2) \cdot P(3|3) \cdot P(2|3) \cdot P(-|2) \\ &= 0.33 \cdot 0.5 \cdot 0.75 \cdot 0.25 \cdot 0.25 \cdot 0.25 \\ &= 0.001933594 \end{aligned}$$

and

$$\begin{aligned} P(HELLO|1-2-3-3-2) &= P(H|1) \cdot P(E|2) \cdot P(L|3) \cdot P(L|3) \cdot P(O|2) \\ &= 0.8 \cdot 0.8 \cdot 0.3 \cdot 0.3 \cdot 0.2 \\ &= 0.01152. \end{aligned}$$

(c) Which sequence $(s_1, s_2, \ldots, s_k)$ with $s_i \in A$ explains the observation $O_1 = \{H, E, L, L, O\}$ best?

$$\begin{aligned} max\{P(1-2-3-3-2) &\cdot P(HELLO|1-2-3-3-2), \\ P(1-2-3-3-3) &\cdot P(HELLO|1-2-3-3-3), \\ P(3-2-3-3-2) &\cdot P(HELLO|3-2-3-3-2), \\ P(3-2-3-3-3) &\cdot P(HELLO|3-2-3-3-3)\} = 3.34125 \cdot 10^{-5} \end{aligned}$$

The maximum value is from sequence $1-2-3-3-3$.

Assignment 8-3 *HMM: Evaluation / Detection*

The Hidden Markov Model (HMM) $M = \{S, B, D, F\}$ with $S = \{A, B, C\}$, $B = \{\clubsuit, \heartsuit, \spadesuit\}$ is given as follows.

$$D = \begin{array}{c} \\ - \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cccc} - & A & B & C \end{array} \\ \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 & 0 \end{pmatrix} \end{array} \quad F = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} \clubsuit & \heartsuit & \spadesuit \end{array} \\ \begin{pmatrix} 1/4 & 3/4 & 0 \\ 0 & 0 & 1 \\ 0 & 1/4 & 3/4 \end{pmatrix} \end{array}$$

(a) Compute the probability of the observation $\clubsuit, \heartsuit, \spadesuit$ without algorithmic procedures. Tag the most probable sequence of states for the observation.

(b) Compute the probability of the observation $\clubsuit, \heartsuit, \spadesuit$ inductively by using the forward-variable

$$\alpha_j(1) = d_{-,j} f_{j,o_1} \quad \alpha_j(t+1) = \left(\sum_{i=1}^{|S|} \alpha_i(t) d_{i,j} \right) f_{j,o_{t+1}}$$

(c) Determine which sequence of states most probably produces the observation $\clubsuit, \heartsuit, \spadesuit$ by using the Viterbi-Algorithm.

$$\delta_j(1) = d_{-,j} f_{j,o_1} \quad \delta_j(t+1) = \left(\max_i \delta_i(t) d_{i,j} \right) f_{j,o_{t+1}}$$

$$\psi_j(1) = 0 \quad \psi_j(t+1) = \arg\max_i \delta_i(t) d_{i,j}$$

(a)

$$\begin{aligned} P(\clubsuit, \heartsuit, \spadesuit) &= P(AAB)P(\clubsuit, \heartsuit, \spadesuit \mid AAB) + P(AAC)P(\clubsuit, \heartsuit, \spadesuit \mid AAC) + P(ACB)P(\clubsuit, \heartsuit, \spadesuit \mid ACB) \\ &= (\frac{1}{3}\frac{1}{4}\frac{1}{4}\frac{1}{4})(\frac{1}{4}\frac{1}{3}1) + (\frac{1}{3}\frac{1}{4}\frac{1}{4}\frac{1}{4})(\frac{1}{4}\frac{3}{4}\frac{3}{4}) + (\frac{1}{3}\frac{1}{4}\frac{1}{2}\frac{1}{4})(\frac{1}{4}\frac{1}{4}1) \\ &= 1/192 \cdot 3/16 + 1/192 \cdot 9/64 + 1/96 \cdot 1/16 = 1/1024 + 3/4096 + 1/1536 \\ &= 29/12288 \approx 0.00236 \end{aligned}$$

The most probable state sequence for the observation is AAB. Note: ACC is not a valid sequence since the transition probability to get from C to C is zero.

(b)

$$\begin{aligned} \alpha_A(1) &= P(A|-) \cdot P(\clubsuit|A) = 1/3 \cdot 1/4 = 1/12 \\ \alpha_B(1) &= P(B|-) \cdot P(\clubsuit|B) = 0 \\ \alpha_C(1) &= P(C|-) \cdot P(\clubsuit|C) = 0 \\ \hline \alpha_A(2) &= (\alpha_A(1) \cdot P(A|A) + \alpha_B(1) \cdot P(A|B) + \alpha_C(1) \cdot P(A|C)) \cdot P(\heartsuit|A) \\ &= (1/12 \cdot 1/4 + 0 \cdot 1/4 + 0 \cdot 1/4) \cdot 3/4 = 1/64 \\ \alpha_B(2) &= (\alpha_A(1) \cdot P(B|A) + \alpha_B(1) \cdot P(B|B) + \alpha_C(1) \cdot P(B|C)) \cdot P(\heartsuit|B) \\ &= (1/12 \cdot 1/4 + 0 \cdot 1/4 + 0 \cdot 1/2) \cdot 0 = 0 \\ \alpha_C(2) &= (\alpha_A(1) \cdot P(C|A) + \alpha_B(1) \cdot P(C|B) + \alpha_C(1) \cdot P(C|C)) \cdot P(\heartsuit|C) \\ &= (1/12 \cdot 1/4 + 0 \cdot 1/2 + 0 \cdot 0) \cdot 1/4 = 1/192 \\ \hline \alpha_A(3) &= (\alpha_A(2) \cdot P(A|A) + \alpha_B(2) \cdot P(A|B) + \alpha_C(2) \cdot P(A|C)) \cdot P(\spadesuit|A) \\ &= (1/64 \cdot 1/4 + 0 \cdot 1/4 + 1/192 \cdot 1/4) \cdot 0 = 0 \\ \alpha_B(3) &= (\alpha_A(2) \cdot P(B|A) + \alpha_B(2) \cdot P(B|B) + \alpha_C(2) \cdot P(B|C)) \cdot P(\spadesuit|B) \\ &= (1/64 \cdot 1/4 + 0 \cdot 1/4 + 1/192 \cdot 1/2) \cdot 1 = 5/786 \\ \alpha_C(3) &= (\alpha_A(2) \cdot P(C|A) + \alpha_B(2) \cdot P(C|B) + \alpha_C(2) \cdot P(C|C)) \cdot P(\spadesuit|C) \\ &= (1/64 \cdot 1/4 + 0 \cdot 1/2 + 1/192 \cdot 0) \cdot 3/4 = 3/1024 \\ \hline &\Rightarrow P(\clubsuit, \heartsuit, \spadesuit) = 5/786 \cdot 1/4 + 3/1024 \cdot 1/4 = 29/12288 \end{aligned}$$

(c)

$$
\begin{array}{ll}
\delta_A(1) = P(A|-) \cdot P(\clubsuit|A) = 1/12 & \psi_A(1) = 0 \\
\delta_B(1) = P(B|-) \cdot P(\clubsuit|B) = 0 & \psi_B(1) = 0 \\
\delta_C(1) = P(C|-) \cdot P(\clubsuit|C) = 0 & \psi_C(1) = 0 \\
\hline
\delta_A(2) = \max\{\delta_A(1) \cdot P(A|A), \delta_B(1) \cdot P(A|B), \delta_C(1) \cdot P(A|C)\} \cdot P(\heartsuit|A) & \\
\qquad = \max\{1/12 \cdot 1/4, 0 \cdot 0, 0 \cdot 1/4\} \cdot 3/4 = 1/64 & \psi_A(2) = A \\
\delta_B(2) = \max\{\delta_A(1) \cdot P(B|A), \delta_B(1) \cdot P(B|B), \delta_C(1) \cdot P(B|C)\} \cdot P(\heartsuit|B) & \\
\qquad = \max\{1/12 \cdot 1/4, 0 \cdot 1/4, 0 \cdot 1/2\} \cdot 0 = 0 & \psi_B(2) = A \\
\delta_C(2) = \max\{\delta_A(1) \cdot P(C|A), \delta_B(1) \cdot P(C|B), \delta_C(1) \cdot P(C|C)\} \cdot P(\heartsuit|C) & \\
\qquad = \max\{1/12 \cdot 1/4, 0 \cdot 1/2, 0 \cdot 0\} \cdot 1/4 = 1/12 \cdot 1/4 \cdot 1/4 = 1/192 & \psi_C(2) = A \\
\hline
\delta_A(3) = \max\{\delta_A(2) \cdot P(A|A), \delta_B(2) \cdot P(A|B), \delta_C(2) \cdot P(A|C)\} \cdot P(\spadesuit|A) & \\
\qquad = \max\{1/64 \cdot 1/4, 0 \cdot 1/4, 1/192 \cdot 1/4\} \cdot 0 = 0 & \psi_A(3) = A \\
\delta_B(3) = \max\{\delta_A(2) \cdot P(B|A), \delta_B(2) \cdot P(B|B), \delta_C(2) \cdot P(B|C)\} \cdot P(\spadesuit|B) & \\
\qquad = \max\{1/64 \cdot 1/4, 0 \cdot 1/4, 1/192 \cdot 1/2\} \cdot 1 = 1/256 & \psi_B(3) = A \\
\delta_C(3) = \max\{\delta_A(2) \cdot P(C|A), \delta_B(2) \cdot P(C|B), \delta_C(2) \cdot P(C|C)\} \cdot P(\spadesuit|C) & \\
\qquad = \max\{1/64 \cdot 1/4, 0 \cdot 1/2, 1/192 \cdot 0\} \cdot 3/4 = 3/1024 & \psi_C(3) = A \\
\hline
\delta_-(4) = \max\{\delta_A(3) \cdot P(-|A), \delta_B(3) \cdot P(-|B), \delta_C(3) \cdot P(-|C)\} & \\
\qquad = \max\{0 \cdot 1/4, 1/256 \cdot 1/4, 3/1024 \cdot 1/4\} \approx \max\{0.0, 0.0009766, 0.0007324\} & \psi(4) = B \\
\qquad \Rightarrow \arg\max P(S_1 S_2 S_3 \mid \clubsuit, \heartsuit, \spadesuit) = AAB &
\end{array}
$$

We can read the results from our ψ values. $\psi(4) = B$ and therefore we get the previous state by checking $\psi_B(3)$ which is A. Again, we can get the previous state by checking $\psi_A(2)$, and so on. We do this until we reach the beginning.