

Exercise 10-3 *ELO Scores*

On a player vs player competition ELO Ranking is used to rate the players. Assume $\alpha = 0.07$ and $\beta = 500$. The following players and their ELO Scores are given:

- Player1: 1000
- Player2: 1200
- Player3: 800

During the competition the following outcomes occur in the given order:

- 1 beats 3
- 2 beats 1
- 3 beats 2

(a) Calculate the ELO Scores at the end of the competition.

Let $X_i \sim N(\mu_i, \beta)$, $X_j \sim N(\mu_j, \beta)$, $Y \sim N(\mu_i - \mu_j, \beta)$ be normally distributed random variables.

$$P(X_i > X_j) = P(X_i - X_j > 0)$$

$$= P(Y > 0)$$

$$= 1 - P(Y \leq 0)$$

$$= 1 - \Phi\left(\frac{0 - (\mu_i - \mu_j)}{\beta}\right)$$

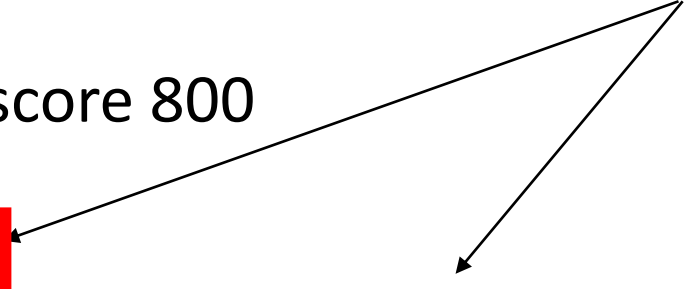
$$= \Phi\left(\frac{\mu_i - \mu_j}{\beta}\right)$$

Update rule:

$$\Delta = \alpha\beta \left(\frac{out + 1}{2} - \Phi\left(\frac{\mu_i - \mu_j}{\beta}\right) \right)$$

Values: $\alpha = 0.07$, $\beta = 500$

First match: p_1 with score 1000 beats p_3 with score 800

$$\Delta_{1v3} = 0.07 * 500 \left(\frac{1 + 1}{2} - \Phi\left(\frac{1000 - 800}{500}\right) \right) = 35 * (1 - 0.65542) = 12.0603$$


\Rightarrow New score for p_1 is $\mu_1 = 1000 + 12 = 1012$

and new score for p_3 is $\mu_3 = 800 - 12 = 788$

Second match: p_2 with score 1200 beats p_1 with score 1012

$$\Delta_{2v1} = 0.07 * 500 \left(\frac{1+1}{2} - \Phi\left(\frac{1200-1012}{500}\right) \right) = 35 * (1-0.64803) = 12.31895$$

\Rightarrow New score for p_2 is $\mu_2 = 1200 + 12 = 1212$
and new score for p_1 is $\mu_1 = 1012 - 12 = 1000$

Third match: p_3 with score 788 beats p_2 with score 1212

$$= 0.07 * 500 \left(\frac{1 + 1}{2} - \Phi\left(\frac{\overset{\Delta_{3v2}}{788 - 1212}}{500}\right) \right) = 35 * (1 - (1 - 0.80234)) = 28.0819$$

\Rightarrow New score for p_3 is $\mu_3 = 788 + 28 = 816$

and new score for p_2 is $\mu_2 = 1212 - 28 = 1184$

(b) Would the ELO Scores differ, if the order of the matches were 2 vs 3, 1 vs 3, 1 vs 2 but with the same results?

$$\Delta_{2v3} = 0.07 * 500 \left(\frac{-1 + 1}{2} - \Phi\left(\frac{1200 - 800}{500}\right) \right) = 35 * (0 - \Phi(0.8)) = -27.585$$

$\Rightarrow \mu_2 = 1173 \text{ and } \mu_3 = 827$

$$\Delta_{1v3} = 0.07 * 500 \left(\frac{1 + 1}{2} - \Phi\left(\frac{1000 - 827}{500}\right) \right) = 35 * (1 - \Phi(0.346)) = 12.71095$$

$\Rightarrow \mu_1 = 1012 \text{ and } \mu_3 = 815$

$$\Delta_{1v2} = 0.07 * 500 \left(\frac{-1 + 1}{2} - \Phi\left(\frac{1012 - 1173}{500}\right) \right) = 35 * (0 - \Phi(-0.322)) = -13.1068$$

$\Rightarrow \mu_2 = 1186 \text{ and } \mu_1 = 999$

Results a):

$$\mu_1 = 1000$$

$$\mu_2 = 1184$$

$$\mu_3 = 816$$

Results b):

$$\mu_1 = 999$$

$$\mu_2 = 1186$$

$$\mu_3 = 815$$