Exercise 10-1  Posterior Distribution

On a day when an assignment is due (A=a), the newsgroup tends to be busy (B=b), and the computer lab tends to be full (C=c). Consider the following conditional probability tables for the domain, where \( A \in \{a, \neg a\} \), \( B \in \{b, \neg b\} \) and \( C \in \{c, \neg c\} \). In this problem, we assume that \( B \) and \( C \) are conditionally independent given \( A \).

\[
\begin{array}{c|c|c}
 & A = a & A = \neg a \\
\hline
B & \frac{0.8}{0.9} & \frac{0.3}{0.7} \\
\hline
\neg b & \frac{0.2}{0.7} & \frac{0.7}{0.3} \\
\end{array}
\quad
\begin{array}{c|c|c}
 & A = a & A = \neg a \\
\hline
C & \frac{0.4}{0.6} & \frac{0.2}{0.8} \\
\hline
\neg c & \frac{0.6}{0.4} & \frac{0.8}{0.2} \\
\end{array}
\]

- Construct the full joint distribution \( P(A,B,C) \) out of these conditional probabilities. Note that this is only possible because of the assumption that \( B \) and \( C \) are conditionally independent given \( A \).

- What is the marginal distribution \( P(B, C) \)? Are these two variables independent in this distribution? Justify your answer using the actual probabilities, not your intuitions.

- What is the posterior distribution over \( A \) given that \( B = b \), that is \( P(A \mid B = b) \)?

- What is the posterior distribution over \( A \) given that \( B = b \) and \( C = c \), that is \( P(A \mid B = b, C = c) \)?

- Briefly explain why the pattern amongst \( P(A) \), \( P(A \mid B = b) \), and \( P(A \mid B = b, C = c) \) makes intuitive sense.
**Exercise 10-2  Bayesian Network**

Consider the following network, in which a mouse agent is reasoning about the behavior of a cat. The mouse really wants to know whether the cat will attack (A), which depends on whether the cat is hungry (H) and whether the cat is sleepy (S). The mouse can observe two things, whether the cat is sleepy (S) and whether the cat has a collar (C). The cat is more often sleepy (S) when it’s either full (f) or starved (v) than when it is peckish (p) and the collar (C) tends to indicate that the cat is not starved. Note that entries are omitted, such as \( P(C = \neg c) \), when their complements are given.

\[
\begin{array}{ccc}
C & P(C) \\
\hline
f & c & 0.7 \\
v & c & 0.1 \\
p & c & 0.2 \\
f & \neg c & 0.2 \\
v & \neg c & 0.5 \\
p & \neg c & 0.3 \\
\end{array}
\]

\[
\begin{array}{ccc}
P(H \mid C) & P(S \mid H) & P(A \mid H, S) \\
\hline
S & H & P \\
\hline
f & s & 0.9 \\
v & s & 0.6 \\
p & s & 0.3 \\
\end{array}
\]

(a) Draw the Bayesian network corresponding to the above joint probability distribution on \( C, H, S, A \).

(b) Compute the following probabilities:

- \( P(A=a, C=c, S=s, H=f) \)
- \( P(A=a, C=c, S=s) \)
- \( P(C=c, S=s) \)
- \( P(A=a \mid C=c, S=s) \)

(c) The mouse is trying to figure out whether it should run out its hole and eat the cheese (E) or do nothing (N). If the mouse hides, nothing happens but it stays hungry. If the mouse runs out to eat the cheese and the cat attacks, the mouse dies (which has a low utility). Otherwise, if the mouse tries to eat the cheese and the cat does not attack, it gets to eat tasty cheese (high utility).

<table>
<thead>
<tr>
<th>Utilities</th>
<th>Cat ready to attack (A)</th>
<th>Mouse’s action</th>
<th>Utility</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>E</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>( \neg a )</td>
<td>E</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Any</td>
<td>N</td>
<td>-2</td>
<td></td>
</tr>
</tbody>
</table>

- Suppose in the above table \( x \) is \(-10\). The mouse sees that the cat has a collar on and is sleepy. What is the utility of trying to eat the cheese? What about doing nothing? Which option should the mouse choose?
- What should the utility of dying (\( x \) in the above table) be in order for the mouse to be ambivalent between running for the cheese and doing nothing? Again, the cat is wearing a collar and is sleepy.

(d) You may have noticed that one of the variables in the network is “collar”, which according to the CPTs (conditional probability table) causes hunger. However, the real relationship is of correlation, not causation. Introduce a new node \( O \) for “owner” and draw a network which better models the true relationship between the variables. \( C \) and \( H \) should be independent conditioned on \( O \).
Exercise 10-3  Markov Property

For each of the following definitions of the state $X_k$ at time $k$ (for $k = 1, 2, \ldots$), determine whether the Markov property is satisfied by the sequence $X_1, X_2, \ldots$

A fair six-sided die (with sides labeled 1,2,3,4,5,6) is rolled repeatedly and independently.

(a) Let $X_k$ denote the largest number obtained in the first $k$ rolls. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

(b) Let $X_k$ denote the number of 6’s obtained in the first $k$ rolls, up to a maximum of ten. (That is, if ten or more 6’s are obtained in the first $k$ rolls, then $X_k = 10$.) Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

(c) Let $Y_k$ denote the result of the $k^{th}$ roll. Let $X_1 = Y_1$ and for $k \geq 2$, let $X_k = Y_k + Y_{k-1}$. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

(d) Let $Y_k = 1$ if the $k^{th}$ roll results is an odd number; and $Y_k = 0$ otherwise. Let $X_1 = Y_1$ and for $k \geq 2$ let $X_k = Y_k \cdot X_{k-1}$. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?