Exercise 6-1  Maximum Likelihood Estimator I

Suppose that $X$ is a discrete random variable with the following probability mass function, where $0 < \theta < 1$ is a parameter.

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td>$2\theta/3$</td>
<td>$\theta/3$</td>
<td>$2(1 - \theta)/3$</td>
<td>$(1 - \theta)/3$</td>
</tr>
</tbody>
</table>

The following 10 independent observations were taken from such a distribution: $(3, 0, 2, 1, 3, 2, 1, 0, 2, 1)$. What is the maximum likelihood estimate of $\theta$?

Exercise 6-2  Maximum Likelihood Estimator II

Suppose when soccer players train penalty kicks, each training session ends after their first miss, since they are demotivated. Let the probability of a miss be $p \in (0, 1)$. Then, the probability for exactly $x_i$ hits ($i \in \{1, \ldots, N\}$) before the first miss can be modeled using the geometric distribution:

$$P(x_i) = p \cdot (1 - p)^{x_i}$$

(a) Following a frequentist approach, determine the maximum likelihood estimator $p_{ML}$ for an i.i.d. (independent identically distributed) population of $N$ training sessions, with hit $x_i \in \{1, \ldots, N\}$ times before their first miss.

(b) Consider the following dataset $X = [7, 2]$ for $N = 2$ training sessions. Compute the probability of a miss.
Exercise 6-3  Frequentist versus Bayesian Statistics

Consider this – rather pathological – example to illustrate the difference between frequentist and bayesian statistics: Alice and Bob play a game in which the first person to get 6 points wins. The points are scored in the following way: A referee is standing at a pool table Alice and Bob cannot see. Before the game begins, the referee rolls a ball onto the table coming to rest at a random position. Each point scored is decided by the referee rolling another ball. If the ball comes to rest left of (the middle of) the initial ball, Alice scores, if it comes to rest right, Bob scores. The players know nothing but who scored a point. If the portion left of the initial ball is denoted as $p$, it is obvious that the probability of Alice scoring a point is $p$.

Now, consider the following situation within the game: Alice has 5 points, Bob has 3. Let us investigate the probability of Bob winning.

(a) Assume that the initial ball came to rest such that $p = 2/3$. What is the probability that Bob wins?

(b) Unfortunately, we do not know $p$ – we only have some data we can try to estimate it from. Follow a frequentist approach: compute the maximum likelihood estimator for $p$ and the probability of Bob winning.

(c) Now, let us follow a bayesian approach: We know that $p$ is only dependent on the position of the initial ball which we assume to be uniformly distributed on the table, i.e., $\text{unif}[0,1]$. Note that we compute the expected probability of Bob winning, as $p$ itself is now drawn from a distribution. Hint: You will need the beta function:

$$B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{(x-1)!(y-1)!}{(x+y-1)!}$$