Linear Algebra (Review)

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2018

Vectors

- k, M, N are scalars
- A one-dimensional array $\mathbf{c}$ is a column vector. Thus in two dimensions,

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

- c_i is the i-th component of $\mathbf{c}$
- $\mathbf{c}^T = (c_1, c_2)$ is a row vector, the transposed of $\mathbf{c}$

Matrices

- A two-dimensional array $\mathbf{A}$ is a matrix, e.g.,

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

- If $\mathbf{A}$ is an $N \times M$-dimensional matrix,
  - then the transposed $\mathbf{A}^T$ is an $M \times N$-dimensional matrix
  - the columns (rows) of $\mathbf{A}$ are the rows (columns) of $\mathbf{A}^T$ and vice versa

$$\mathbf{A}^T = \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \\ a_{1,3} & a_{2,3} \end{bmatrix}$$

Addition of Two Vectors

- Let $\mathbf{c} = \mathbf{a} + \mathbf{d}$
- Then $c_j = a_j + d_j$

Multiplication of a Vector with a Scalar

- $\mathbf{c} = k\mathbf{a}$ is a vector with $c_j = ka_j$
- $\mathbf{C} = k\mathbf{A}$ is a matrix of the dimensionality of $\mathbf{A}$, with $c_{i,j} = ka_{i,j}$

Scalar Product of Two Vectors

- The **scalar product** (also called dot product) is defines as

$$\mathbf{a} \cdot \mathbf{c} = \mathbf{a}^T \mathbf{c} = \sum_{j=1}^{M} a_j c_j$$

  and is a scalar

- Special case:

$$\mathbf{a}^T \mathbf{a} = \sum_{j=1}^{M} a_j^2$$

Matrix-Vector Product

- A matrix consists of many row vectors. So a product of a matrix with a column vector consists of many scalar products of vectors
- If $\mathbf{A}$ is an $N \times M$-dimensional matrix and $\mathbf{c}$ is an M-dimensional column vector
- Then $\mathbf{d} = \mathbf{Ac}$ is an N-dimensional column vector $\mathbf{d}$ with

$$d_i = \sum_{j=1}^{M} a_{i,j} c_j$$

Matrix-Matrix Product

- A matrix also consists of many column vectors. So a product of matrix with a matrix consists of many matrix-vector products
- If $\mathbf{A}$ is an $N \times M$-dimensional matrix and $\mathbf{C}$ an $M \times K$-dimensional matrix
- Then $\mathbf{D} = \mathbf{AC}$ is an $N \times K$-dimensional matrix with

$$d_{i,k} = \sum_{j=1}^{M} a_{i,j} c_{j,k}$$

Multiplication of a Row-Vector with a Matrix

- **Multiplication of a row vector with a matrix is a row vector**. If A is a $N \times M$-dimensional matrix and $\mathbf{d}$ a N-dimensional vector and if

$$\mathbf{c}^T = \mathbf{d}^T A$$

Then $\mathbf{c}$ is a M-dimensional vector with $c_j = \sum_{i=1}^{N} d_i a_{i,j}$

Outer Product

- Special case: **Multiplication of a column vector with a row vector is a matrix**.
  Let $\mathbf{d}$ be a N-dimensional vector and $\mathbf{c}$ be a M-dimensional vector, then

  $$\mathbf{A} = \mathbf{d}\mathbf{c}^T$$

  is an $N \times M$ matrix with $a_{i,j} = d_i c_j$

  Example:

  $$\begin{bmatrix} d_1 c_1 & d_1 c_2 & d_1 c_3 \\ d_2 c_1 & d_2 c_2 & d_2 c_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$$

Matrix Transposed

- The transposed $\mathbf{A}^T$ changes rows and columns
- We have

$$\left(\mathbf{A}^T\right)^T = \mathbf{A}$$

$$(\mathbf{AC})^T = \mathbf{C}^T \mathbf{A}^T$$

Unit Matrix

- $$\mathbf{I} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & & \ldots & \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

Diagonal Matrix

- $N \times N$ diagonal matrix:

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & 0 & \dots & 0 \\ 0 & a_{2,2} & \dots & 0 \\ & & \dots & \\ 0 & \dots & 0 & a_{N,N} \end{pmatrix}$$

Matrix Inverse

- Let $\mathbf{A}$ be an $N \times N$ square matrix
- If there is a unique inverse matrix $\mathbf{A}^{-1}$, then we have

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \quad \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

- If the corresponding inverse exist, $(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}$

Orthogonal Matrices

- **Orthogonal Matrix** (more precisely: Orthonormal Matrix): $\mathbf{R}$ is a (quadratic) orthogonal matrix, if all columns are orthonormal. It follows (non-trivially) that all rows are orthonormal as well and

$$\mathbf{R}^T\mathbf{R} = \mathbf{I} \quad \mathbf{R}\mathbf{R}^T = \mathbf{I} \quad \mathbf{R}^{-1} = \mathbf{R}^T \tag{1}$$

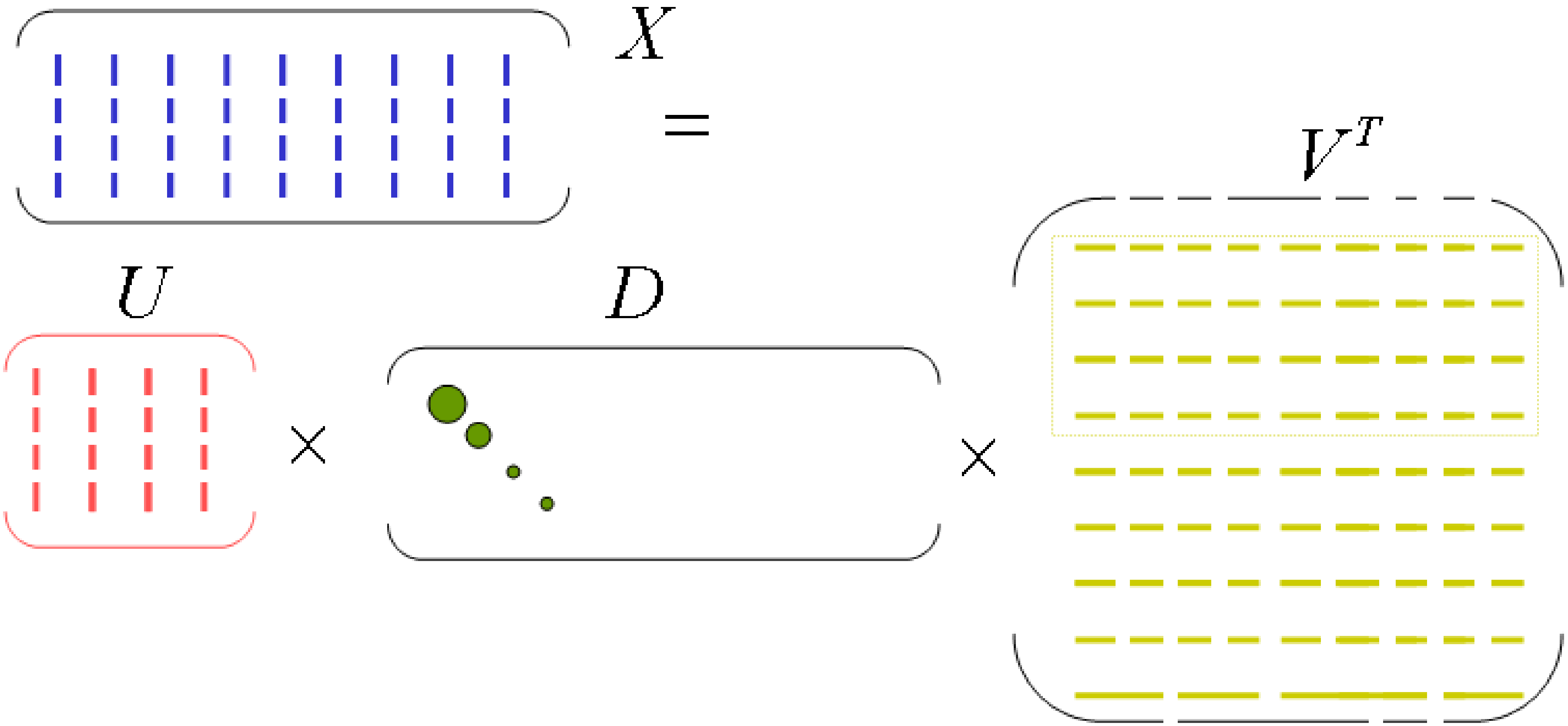
Singular Value Decomposition (SVD)

- Any $N \times M$ matrix $\mathbf{X}$ can be factored as

$$\mathbf{X} = \mathbf{U}\mathbf{D}\mathbf{V}^T$$

where $\mathbf{U}$ and $\mathbf{V}$ are both **orthonormal** matrices. $\mathbf{U}$ is an $N \times N$ Matrix and $\mathbf{V}$ is an $M \times M$ Matrix.

- $\mathbf{D}$ is an $N \times M$ **diagonal matrix** with diagonal entries (singular values) $d_i \geq 0, i = 1, \ldots, \tilde{r}$, with $\tilde{r} = \min(M, N)$
- The $\mathbf{u}_j$ (columns of $\mathbf{U}$) are the left singular vectors
- The $\mathbf{v}_j$ (columns of $\mathbf{V}$) are the right singular vectors
- The d_j (diagonal entries of $\mathbf{D}$) are the singular values



X
$=$
V^T
U
D
$\times$
$\times$

Appendix*

- A vector is defined in a vector space. Example: $\mathbf{c} \in \mathbb{R}^2$ and $\mathbf{c} = c_i \mathbf{e}_1 + c_2 \mathbf{e}_2$ with an orthogonal basis $\mathbf{e}_1, \mathbf{e}_2$. We denote with $\mathbf{c}$ both the vector and its component representation

- A matrix is a 2-D array that is defined with respect to a vector space

- The dot product is identical to the **inner product** $\langle \mathbf{a}, \mathbf{c} \rangle$ for Euclidean vector spaces with orthonormal basis vectors $\mathbf{e}_i$

$$\langle \mathbf{a}, \mathbf{c} \rangle = \left(\sum_i a_i \mathbf{e}_i \right) \left(\sum_{i'} c_{i'} \mathbf{e}_{i'} \right) = \sum_i a_i c_i = \mathbf{a} \cdot \mathbf{c} = \mathbf{a}^T \mathbf{c}$$

- An outer product is also called a dyadic product or an **outer product** (when related to vector spaces) and is written as $\mathbf{d} \otimes \mathbf{c}$. Note that a matrix is generated from two vectors

- An outer product is a special case of a **tensor product**

- $\mathbf{C} = \mathbf{A}\mathbf{B}^T$, can be written as a sum of outer products $\mathbf{C} = \sum_j \mathbf{a}_j \mathbf{b}_j^T$, where $\mathbf{a}_j$ is a columns vector of $\mathbf{A}$ and $\mathbf{b}_j$ is a columns vector of $\mathbf{B}$