Exercise 8-1  Optimal Separating Hyperplane

Consider the following dataset consisting of points \((x_1, x_2)\) in \(\mathbb{R}^2\). Using a hyperplane, points marked by \(\times\) are to be mapped onto \(\geq 1\), points marked by \(\bigcirc\) are to be mapped onto \(\leq -1\).

(a) Find the support vectors.

(b) Determine the equation of one separating hyperplane \(h = x^T w\), optimize it and draw it within the figure.

(c) Compute the margin \(C\).
Exercise 8-2  Determining the Optimal Separating Hyperplane

Determine the optimal separating hyperplane of the following dataset, partitioned into two classes $A$ and $B$:

$$A = \left\{ p_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, p_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, p_3 = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, p_4 = \begin{pmatrix} 2.5 \\ 3 \end{pmatrix}, p_5 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\},$$

$$B = \left\{ p_6 = \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix}, p_7 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, p_4 = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} \right\}$$

Instances of class $A$ shall be labeled with $1$, instances of class $B$ with $-1$.

Name the support vectors, compute the optimal separating hyperplane and visualize the result. How wide is the margin?

Exercise 8-3  Lagrange Multipliers

To find critical points of a function $f(x,y,z)$ on a level surface $g(x,y,z) = C$, we must solve the following system of equations:

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$
$$g(x,y,z) = C$$

Remember that $\nabla f$ and $\nabla g$ are vectors, we can write this as a collection of four equations in the four unknowns $x,y,z$ and $\lambda$:

$$f_x(x,y,z) = \lambda g_x(x,y,z)$$
$$f_y(x,y,z) = \lambda g_y(x,y,z)$$
$$f_z(x,y,z) = \lambda g_z(x,y,z)$$
$$g(x,y,z) = C$$

Once you have found all critical points, you plug them into $f$ to see where the maxima and minima. The critical points where $f$ is greatest are maxima and the critical points where $f$ is smallest are minima.

- Use Lagrange multipliers to find all the critical points of $f$ on the given surface (or curve)
- Determine the maxima and minima of $f$ on the surface (or curve) by evaluating $f$ at the critical values.

(a) The function $f(x,y,z) = x + y + 2z$ on the surface $x^2 + y^2 + z^2 = 3$

(b) The function $f(x,y) = xy$ on the curve $3x^2 + y^2 = 6$