Linear Algebra (Review)

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Vectors

- k is a scalar (a number)
- $\mathbf{c}$ is a column vector. Thus in two dimensions,

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

- (Advanced: More precisely, a vector is defined in a vector space. Example: $\mathbf{c} \in \mathbb{R}^2$ and $\mathbf{c} = c_i\mathbf{e}_1 + c_2\mathbf{e}_2$ with an orthogonal basis $\mathbf{e}_1, \mathbf{e}_2$. We denote with $\mathbf{c}$ both the vector and its component representation)
- c_i is the i-th component of $\mathbf{c}$
- $\mathbf{c}^T = (c_1, c_2)$ is a row vector, the transposed of $\mathbf{c}$

Matrices

- A is a matrix. (Advanced: A matrix is a 2-D array that is defined with respect to a vector space.)
- If A is a $k \times l$-dimensional matrix,
  - then the transposed A^T is an $l \times k$-dimensional matrix
  - the columns (rows) of A are the rows (columns) of A^T and vice versa

Addition of Two Vectors

- Let $\mathbf{c} = \mathbf{a} + \mathbf{d}$
- Then $c_j = a_j + d_j$

Multiplication of a Vector with a Scalar

- $\mathbf{c} = k\mathbf{a}$ is a vector with $c_j = ka_j$
- $C = kA$ is a matrix of the dimensionality of A, with $c_{i,j} = ka_{i,j}$

Scalar Product of Two Vectors

- The **scalar product** (also called dot product) is defines as

$$\mathbf{a} \cdot \mathbf{c} = \mathbf{a}^T \mathbf{c} = \sum_{m=1}^{l} a_m c_m$$

and is a scalar

- Advanced: The dot product is identical to the **inner product** $\langle \mathbf{a}, \mathbf{c} \rangle$ for Euclidean vector spaces with orthonormal basis vectors $\mathbf{e}_i$

$$\langle \mathbf{a}, \mathbf{c} \rangle = \left(\sum_i a_i \mathbf{e}_i \right) \left(\sum_{i'} c_{i'} \mathbf{e}_{i'} \right) = \sum_i a_i c_i = \mathbf{a} \cdot \mathbf{c} = \mathbf{a}^T \mathbf{c}$$

Matrix-Vector Product

- A matrix consists of many row vectors. So a product of a matrix with a column vector consists of many scalar products of vectors
- If A is a $k \times l$-dimensional matrix and $\mathbf{c}$ a l-dimensional column vector
- Then $\mathbf{d} = A\mathbf{c}$ is a k-dimensional column vector $\mathbf{d}$ with

$$d_j = \sum_{m=1}^{l} a_{j,m} c_m$$

Matrix-Matrix Product

- A matrix also consists of many column vectors. So a product of matrix with a matrix consists of many matrix-vector products
- If A is a $k \times l$-dimensional matrix and C an $l \times p$-dimensional matrix
- Then $D = AC$ is a $k \times p$-dimensional matrix with

$$d_{i,j} = \sum_{m=1}^{l} a_{i,m} c_{m,j}$$

Multiplication of a Row-Vector with a Matrix

- **Multiplication of a row vector with a matrix is a row vector**. If A is a $k \times l$-dimensional matrix and $\mathbf{d}$ a k-dimensional Vector and if

$$\mathbf{c}^T = \mathbf{d}^T A$$

Then $\mathbf{c}$ is a l-dimensional vector with $c_i = \sum_{m=1}^{k} d_m a_{m,i}$

Outer Product

- Special case: **Multiplication of a column vector with a row vector is a matrix**. Let $\mathbf{d}$ be a k-dimensional vector and $\mathbf{c}$ be a p-dimensional vector, then

$$A = \mathbf{d}\mathbf{c}^T$$

is a $k \times p$ matrix with $a_{i,j} = d_i c_j$. Advanced: This is also called an **outer product** (when related to vector spaces) and is written as $\mathbf{d} \otimes \mathbf{c}$. Note that a matrix is generated from two vectors

- Advanced: An outer product is a special case of a **tensor product**

Matrix Transposed

- The transposed A^T changes rows and columns
- $$\left(A^T\right)^T = A$$
- $$(AC)^T = C^T A^T$$

Unit Matrix

- $$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & & \dots & \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

Diagonal Matrix

- $k \times k$ diagonal matrix:

$$A = \begin{pmatrix} a_{1,1} & 0 & \ldots & 0 \\ 0 & a_{2,2} & \ldots & 0 \\ & & \ldots & \\ 0 & \ldots & 0 & a_{k,k} \end{pmatrix}$$

Matrix Inverse

- Let A be a square matrix
- If there is a unique inverse matrix A^{-1}, then we have

$$A^{-1}A = I \quad AA^{-1} = I$$

- If the corresponding inverse exist, $(AC)^{-1} = C^{-1}A^{-1}$

Orthogonal Matrices

- **Orthogonal Matrix (more precisely: Orthonormal Matrix)**: R is a (quadratic) orthogonal matrix, if all columns are orthonormal. It follows (non-trivially) that all rows are orthonormal as well and

$$R^T R = I \quad RR^T = I \quad R^{-1} = R^T \tag{1}$$

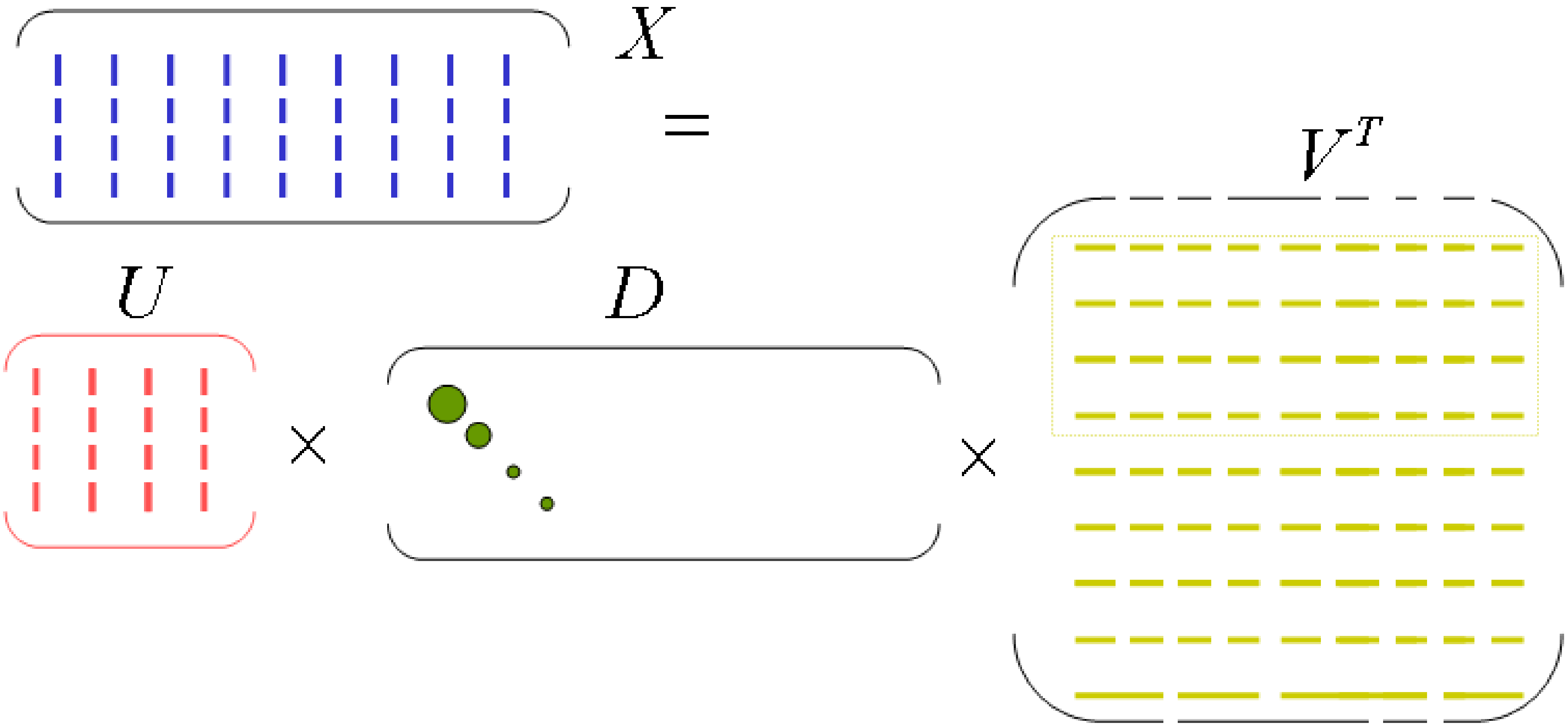
Singular Value Decomposition (SVD)

- Any $N \times M$ matrix X can be factored as

$$X = UDV^T$$

  where U and V are both **orthonormal** matrices. U is an $N \times N$ Matrix and V is an $M \times M$ Matrix.

- D is an $N \times M$ **diagonal matrix** with diagonal entries (singular values) $d_i \geq 0, i = 1, \ldots, \tilde{r}$, with $\tilde{r} = \min(M, N)$
- The $\mathbf{u}_j$ (columns of U) are the left singular vectors
- The $\mathbf{v}_j$ (columns of V) are the right singular vectors
- The d_j (diagonal entries of D) are the singular values



X
$=$
V^T
U
D
$\times$
$\times$