Linear Algebra (Review)

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Vectors

• $k$ is a scalar

• $\mathbf{c}$ is a column vector. Thus in two dimensions,

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

• (More precisely, a vector is defined in a vector space. Example: $\mathbf{c} \in \mathbb{R}^2$ and $\mathbf{c} = c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2$ with an orthogonal basis $\mathbf{e}_1, \mathbf{e}_2$. We denote with $\mathbf{c}$ both the vector and its component representation)

• $c_i$ is the $i$-th component of $\mathbf{c}$

• $\mathbf{c}^T = (c_1, c_2)$ is a row vector, the transposed of $\mathbf{c}$
Matrices

• $A$ is a matrix. (A matrix is a 2-D array that is defined with respect to a vector space.)

• If $A$ is a $k \times l$-dimensional matrix,
  
  – then the transposed $A^T$ is an $l \times k$-dimensional matrix

  – the columns (rows) of $A$ are the rows (columns) of $A^T$ and vice versa
Addition of Two Vectors

- Let \( c = a + d \)
- Then \( c_j = a_j + d_j \)
Multiplication of a Vector with a Scalar

- \( c = ka \) is a vector with \( c_j = ka_j \)
- \( C = kA \) is a matrix of the dimensionality of \( A \), with \( c_{i,j} = ka_{i,j} \)
Scalar Product of Two Vectors

- The **scalar product** (also called dot product) is defined as

\[
a \cdot c = a^T c = \sum_{m=1}^{l} a_m c_m
\]

and is a scalar.

- The dot product is identical to the **inner product** \(\langle a, c \rangle\) for Euclidean vector spaces with orthonormal basis vectors \(e_i\)

\[
\langle a, c \rangle = \left( \sum_i a_i e_i \right) \left( \sum_{i'} c_{i'} e_{i'} \right) = \sum_i a_i c_i = a \cdot c = a^T c
\]
Matrix-Vector Product

- A matrix consists of many row vectors. So a product of a matrix with a column vector consists of many scalar products of vectors.

- If $A$ is a $k \times l$-dimensional matrix and $c$ a $l$-dimensional column vector.

- Then $d = Ac$ is a $k$-dimensional column vector $d$ with

$$d_j = \sum_{m=1}^{l} a_{j,m} c_m$$
Matrix-Matrix Product

• A matrix also consists of many column vectors. So a product of matrix with a matrix consists of many matrix-vector products

• If $A$ is a $k \times l$-dimensional matrix and $C$ an $l \times p$-dimensional matrix

• Then $D = AC$ is a $k \times p$-dimensional matrix with

\[
d_{i,j} = \sum_{m=1}^{l} a_{i,m} c_{m,j}
\]
Multiplication of a Row-Vector with a Matrix

- Multiplication of a row vector with a matrix is a row vector. If $A$ is a $k \times l$-dimensional matrix and $d$ a $k$-dimensional Vector and if

$$c^T = d^T A$$

Then $c$ is a $l$-dimensional vector with $c_i = \sum_{m=1}^{k} d_m a_{m,i}$
Outer Product

- Special case: **Multiplication of a column vector with a row vector is a matrix.**
  Let \( \mathbf{d} \) be a \( k \)-dimensional vector and \( \mathbf{c} \) be a \( p \)-dimensional vector, then

  \[
  A = \mathbf{d}\mathbf{c}^T
  \]

  is a \( k \times p \) matrix with \( a_{i,j} = d_i c_j \). This is also called an **outer product** (when related to vector spaces) and is written as \( \mathbf{d} \otimes \mathbf{c} \). Note that a matrix is generated from two vectors.

- An outer product is a special case of a **tensor product**
Matrix Transposed

- The transposed $A^T$ changes rows and columns

- $(A^T)^T = A$

- $(AC)^T = C^T A^T$
Unit Matrix

\[ I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \]
Diagonal Matrix

- $k \times k$ diagonal matrix:

$$A = \begin{pmatrix}
a_{1,1} & 0 & \cdots & 0 \\
0 & a_{2,2} & \cdots & 0 \\
& \ddots & \ddots & \ddots \\
0 & \cdots & 0 & a_{k,k}
\end{pmatrix}$$
Matrix Inverse

• Let $A$ be a square matrix

• If there is a unique inverse matrix $A^{-1}$, then we have

$$A^{-1}A = I \quad AA^{-1} = I$$

• If the corresponding inverse exist, $(AC)^{-1} = C^{-1}A^{-1}$
Orthogonal Matrices

- **Orthogonal Matrix (more precisely: Orthonormal Matrix):** \( R \) is a (quadratic) orthogonal matrix, if all columns are orthonormal. It follows (non-trivially) that all rows are orthonormal as well and

\[
R^T R = I \quad RR^T = I \quad R^{-1} = R^T \tag{1}
\]
Singular Value Decomposition (SVD)

• Any $N \times M$ matrix $X$ can be factored as

$$X = UDV^T$$

where $U$ and $V$ are both orthonormal matrices. $U$ is an $N \times N$ Matrix and $V$ is an $M \times M$ Matrix.

• $D$ is an $N \times M$ diagonal matrix with diagonal entries (singular values) $d_i \geq 0$, $i = 1, \ldots, \tilde{r}$, with $\tilde{r} = \min(M, N)$

• The $u_j$ (columns of $U$) are the left singular vectors

• The $v_j$ are the right singular vectors

• The $d_j$ are the singular values
\[ X = U D V^T \]