Linear Algebra (Review)

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Vectors

- k is a scalar
- $\mathbf{b}$ is a column vector
- b_i is the i-th component of $\mathbf{b}$
- $\mathbf{b}^T$ is a row vector, the transposed of $\mathbf{b}$

Matrices

- A is a matrix
- If A is a $k \times l$-dimensional matrix,
  - then the transposed A^T is an $l \times k$-dimensional matrix
  - the columns (rows) of A are the rows (columns) of A^T ad vice versa

Multiplication with a Scalar

- $\mathbf{c} = k\mathbf{b}$ is a vector with dimensionality $\mathbf{b}$, with $c_i = kb_i$
- $C = kA$ is a matrix of the dimensionality of A, with $c_{i,j} = ka_{i,j}$

(Inner) Product of two Vectors

$$\mathbf{b}^T\mathbf{c} = \sum_{m=1}^{l} b_m c_m$$

is a scalar

Matrix-Vector Product

- A matrix consists of many row vectors. So a product of matrix with a vector consists of many inner products of vectors
- If A is a $k \times l$-dimensional matrix and $\mathbf{b}$ a l-dimensional column vector, then
- Then $\mathbf{c} = A\mathbf{b}$ is a k-dimensional column vector with

$$c_i = \sum_{m=1}^{l} a_{i,m} b_m$$

Matrix-Matrix Product

- A matrix consists of many column vectors. So a product of matrix with a matrix consists of many matrix-vector products
- If A is a $k \times l$-dimensional matrix and C a $l \times p$-dimensional matrix
- Then $D = AC$ is a $k \times p$-dimensional matrix with

$$d_{i,j} = \sum_{m=1}^{l} a_{i,m} c_{m,j}$$

Matrix-Matrix Product (cont'd)

- Special case: **Multiplication of a row vector with a column vector is a scalar**. If $\mathbf{b}$ and $\mathbf{c}$ are two $l-$dimensional row vectors, then

$$\mathbf{b}^T\mathbf{c} = \sum_{m=1}^{l} b_m c_m$$

- Special case: **Multiplication of a column vector with a row vector is a matrix**. Is $\mathbf{b}$ a k-dimensional vector and $\mathbf{c}$ is a p-dimensional vector, then

$$A = \mathbf{b}\mathbf{c}^T$$

is a $k \times p$ matrix with $a_{i,j} = b_i c_j$

Multiplication of two Matrices(cont'd)

- Special case: **Multiplication of a Matrix with a column vector is a column vector**. If A is a $k \times l$-dimensional matrix and is $\mathbf{b}$ an l-dimensional vector, then

  $$\mathbf{c} = A\mathbf{b}$$

  is a k-dimensional vector with t $c_i = \sum_{m=1}^{l} a_{i,m} b_m$

- Special case: **Multiplication of a row vector with a matrix is a row vector**. If A is a $k \times l$-dimensional matrix and $\mathbf{b}$ a k-dimensional Vector and if

  $$\mathbf{c}^T = \mathbf{b}^T A$$

  Then $\mathbf{c}$ is a l-dimensional vector with $c_i = \sum_{m=1}^{k} b_m a_{m,i}$

Matrix Transposed

- The transposed A^T changes rows and columns
- $$\left(A^T\right)^T = A$$
- $$(AB)^T = B^T A^T$$

Unit Matrix

- $$I = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & & \ldots & \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

Matrix Inverse

- Let A be a square matrix
- If there is a unique inverse matrix A^{-1}, then we have

$$A^{-1}A = I \quad AA^{-1} = I$$

- If the corresponding inverse exist, $(AB)^{-1} = B^{-1}A^{-1}$

Orthogonal Matrices

- **Orthogonal Matrix (more precisely: Orthonormal Matrix)**: R is a (quadratic) orthogonal matrix, if all columns are orthonormal. It follows (non-trivially) that all rows are orthonormal as well and

$$R^T R = I \qquad RR^T = I \qquad R^{-1} = R^T \tag{1}$$