Chapter 8:
Graph Data

Part 2:
Community Detection

Based on
Leskovec, Rajaraman, Ullman 2014:
Mining of Massive Datasets

Outline

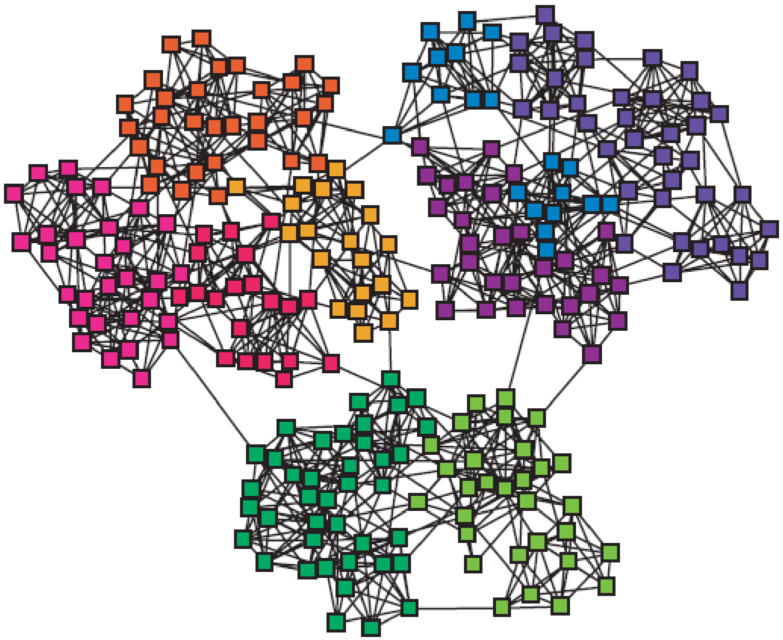
Community Detection

- Social networks
- Betweenness
  - Girvan-Newman Algorithm
- Modularity
- Graph Partitioning
  - Spectral Graph Partitioning
- Trawling

Networks & Communities:

Think of networks being organized into:

- **Modules**
- **Cluster**
- **Communities**



→ **Goal: Find densely linked clusters**

What is a Social Network?

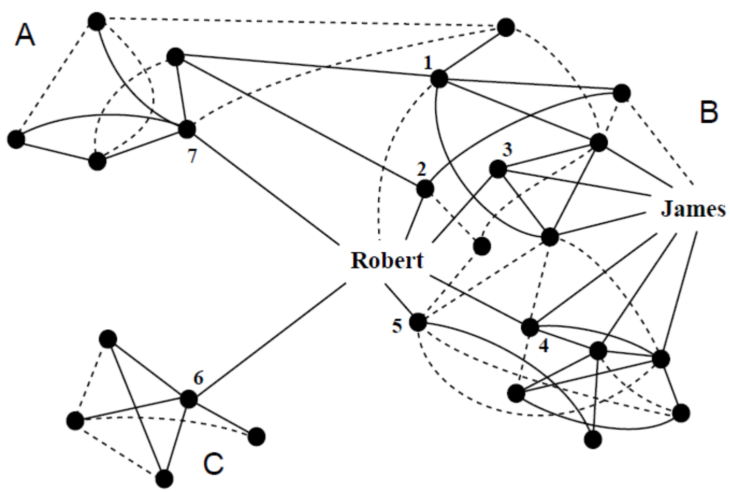
Characteristics of a social network:

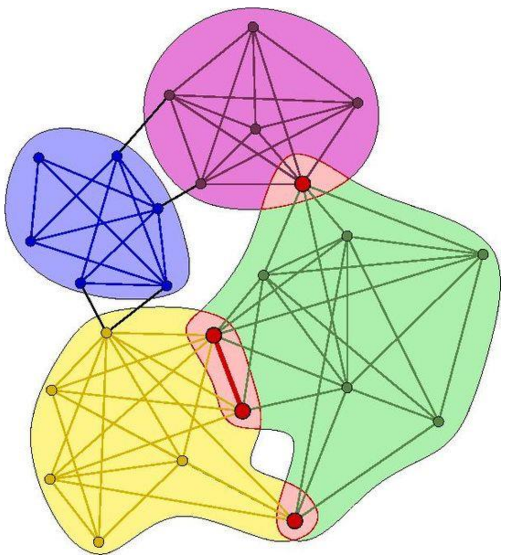
- **Collection of entities** participating in the network (entities might be individuals, phone numbers, email addresses , …)

- At least one **relationship between entities** of the network. (Facebook: 'friend'). Relationship can be all-or-nothing or specified by a degree (e.g. fraction of the average day that two people communication to each other)

- Assumption of **nonrandomness** or **locality**, i.e. relationships tend to cluster. (e.g. A is related to B and C → higher probability that B is related to C)

How to find communities?





→ Here we will work with undirected (unweighted networks)

→ We need to resolve 2 questions:

  → How to compute betweenness?

  → How to select the number of clusters?

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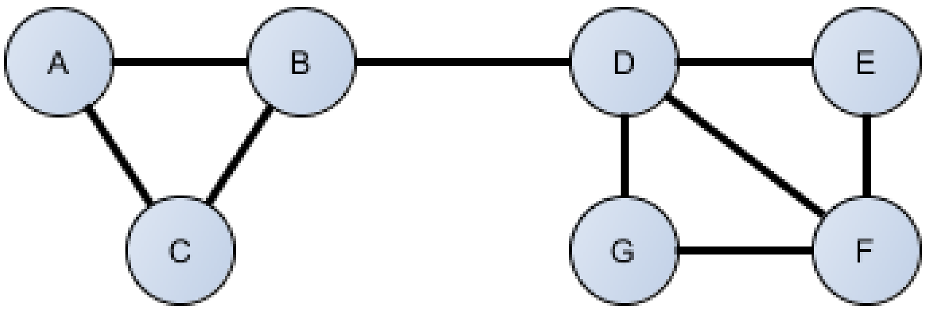
Betweenness

Definition: The betweenness of an edge (a,b) is the number of pairs of nodes x and y such that (a,b) lies on the shortest-path between x and y.



Example:
Edge (B,D) has highest betweenness (shortest path of A,B,C to any of D,E,F,G)
→ Betweenness of (B,D) aggregates to: 3 x 4 = 12

Betweenness of edge (D,F) ?

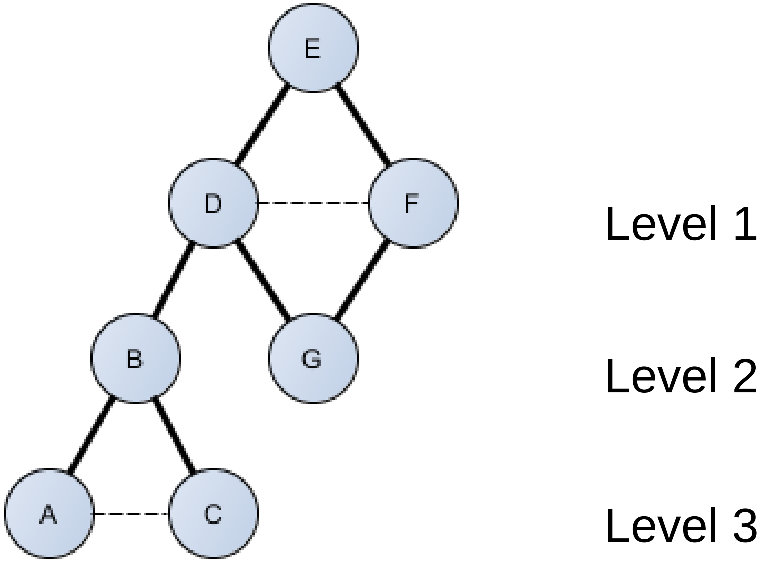
Girvan-Newman Algorithm

Goal: Computation of betweenness of edges

Step 1: Perform a breadth-first search, starting at node X and construct a DAG (directed, acyclic graph)

Example:

- Start at node E

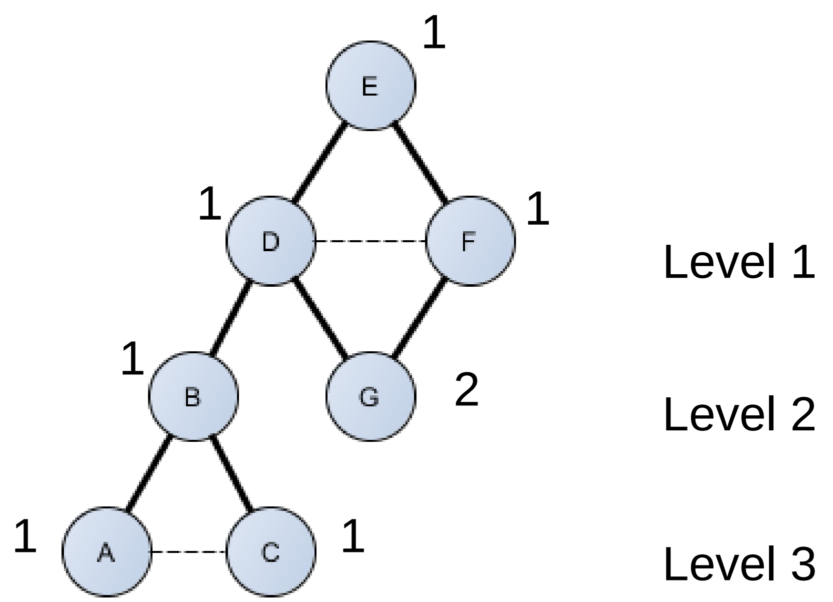


Girvan-Newman Algorithm

Goal: Computation of betweenness of edges

Step 2: label each node by the number of shortest paths that reach it from the root. Label of root = 1, each node is labeled by the sum of its parents.

Example:



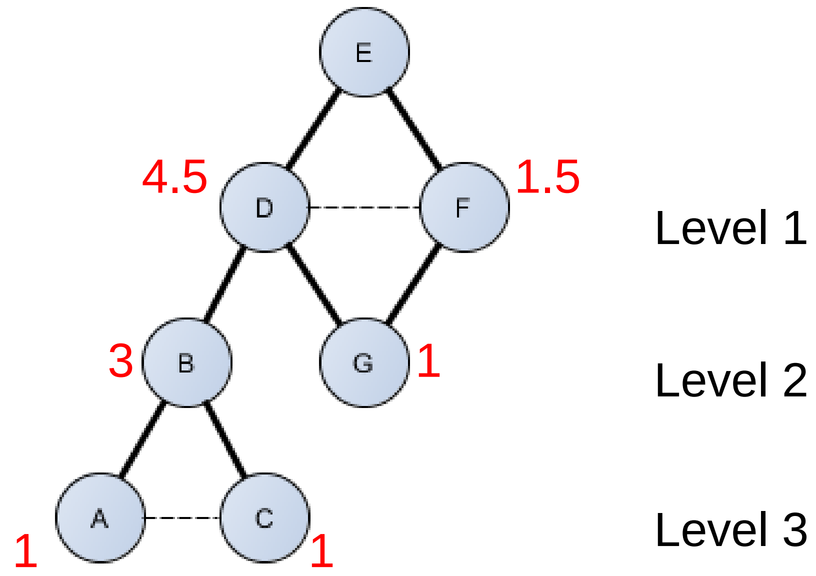
Girvan-Newman Algorithm

Goal: Computation of betweenness of edges

Step 3: calculate for each edge e the sum over all nodes Y the fraction of shortest paths from the root X to Y:

In Detail:

1. Each leaf gets a credit of 1.
2. Non-leaf nodes gets a credit of 1 plus the sum of its children
3. A DAG edge e entering node Z from the level above is given a share of the credit of Z proportional to the fraction of shortest paths from the root to Z. **Formally:** let $Y_1, \dots, Y_k$ be the parent nodes of Z with $p_i, 1 \leq i \leq k$, be the number of shortest path to Y_i. The credit for edge (Y_i, Z) is given by: $Z * p_i \; / \; \sum_{j=1}^{k} p_j$

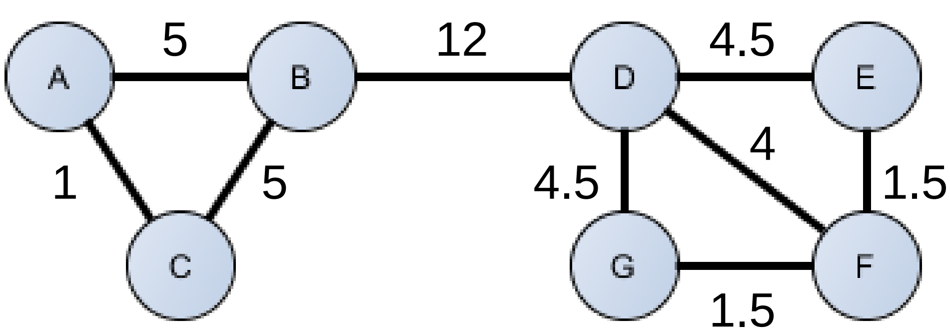


Find Communities using Betweenness

Idea: Clustering is performed by taking the edges in order of increasing betweenness and add them to the graph or as a process of edge removal

Example:
GN-Algorithm has been performed for every node and the credit of each edge has been calculated (by summing the credits up and dividing them by 2. Why?)

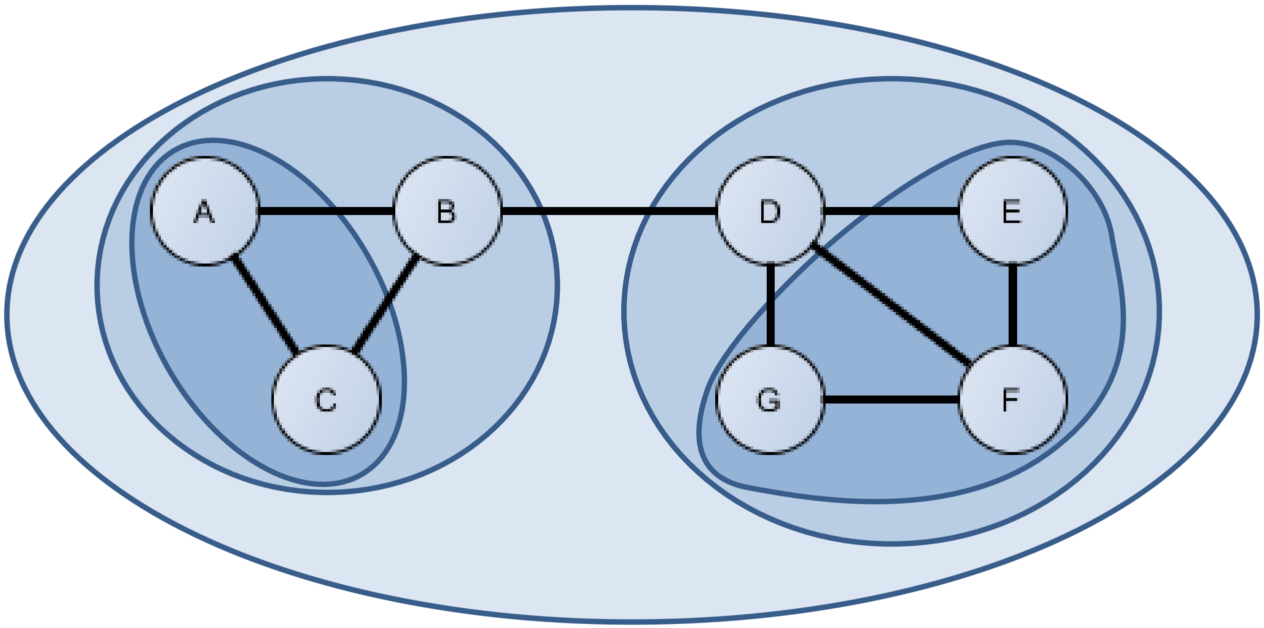
Remove edges, starting by highest betweenness:
- 1. Remove (B,D)
  → Communities {A,B,C} and {D,E,F,G}
- 2. Remove (A,B), (B,C), (D,G), (D,E), (D,F)
  → Communities {A,C} and {E,F,G}
  Node B and D are encapsulated as ‚traitors' of communities



Find Communities using Betweenness

Girvan-Newman Algorithm:

- connected components are communities
- gives a hierarchical decomposition of the network



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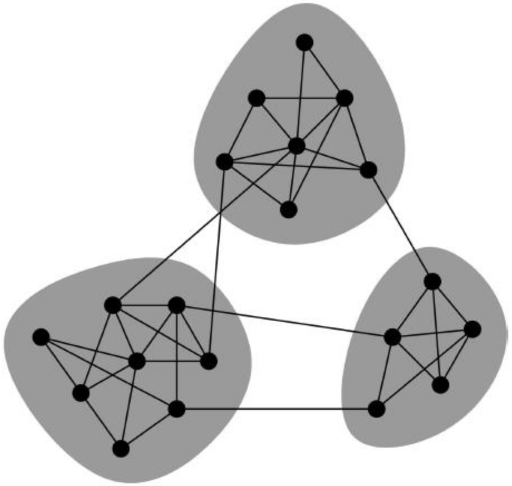
Network Communities

- ✓ How to compute betweenness?
- → **How to select the number of clusters?**



Communities: sets of tightly connected nodes

Modularity Q:

- A measure of how well a network is partitioned into communities
- Given a partitioning of the network into groups $s \in S$:

$$\boldsymbol{Q} \propto \sum_{s \in S} [(\#\boldsymbol{edges\ within\ group\ s}) - \underbrace{(\boldsymbol{expected\ \#edges\ within\ group\ s})}_{\text{Defined by null model}}]$$

Null Model: Configuration Model

- Given a graph G with $\boldsymbol{n}$ **nodes** and $\boldsymbol{m}$ **edges**, construct rewired network G':
  - Same degree distribution but random connections
  - Consider G' as a **multigraph**

→ The **expected number of edges between nodes $\boldsymbol{i}$ and $\boldsymbol{j}$** of degrees k_i and k_j is given by: $\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{m}} * \boldsymbol{k_i k_j}$

Proof that G' contains the expected number of m edges:

$$\frac{1}{2}\sum_{i\in N}\sum_{j\in N}\frac{k_i k_j}{2m} = \frac{1}{2}\frac{1}{2m}\sum_{i\in N} k_i \left(\sum_{j\in N} k_j\right) = \frac{1}{4m} * 2m * 2m = m$$

Modularity

Modularity of partitioning S of graph G:

$$Q \propto \sum_{s \in S} [(\#edges\ within\ group\ s) - (expected\ \#edges\ within\ group\ s)]$$

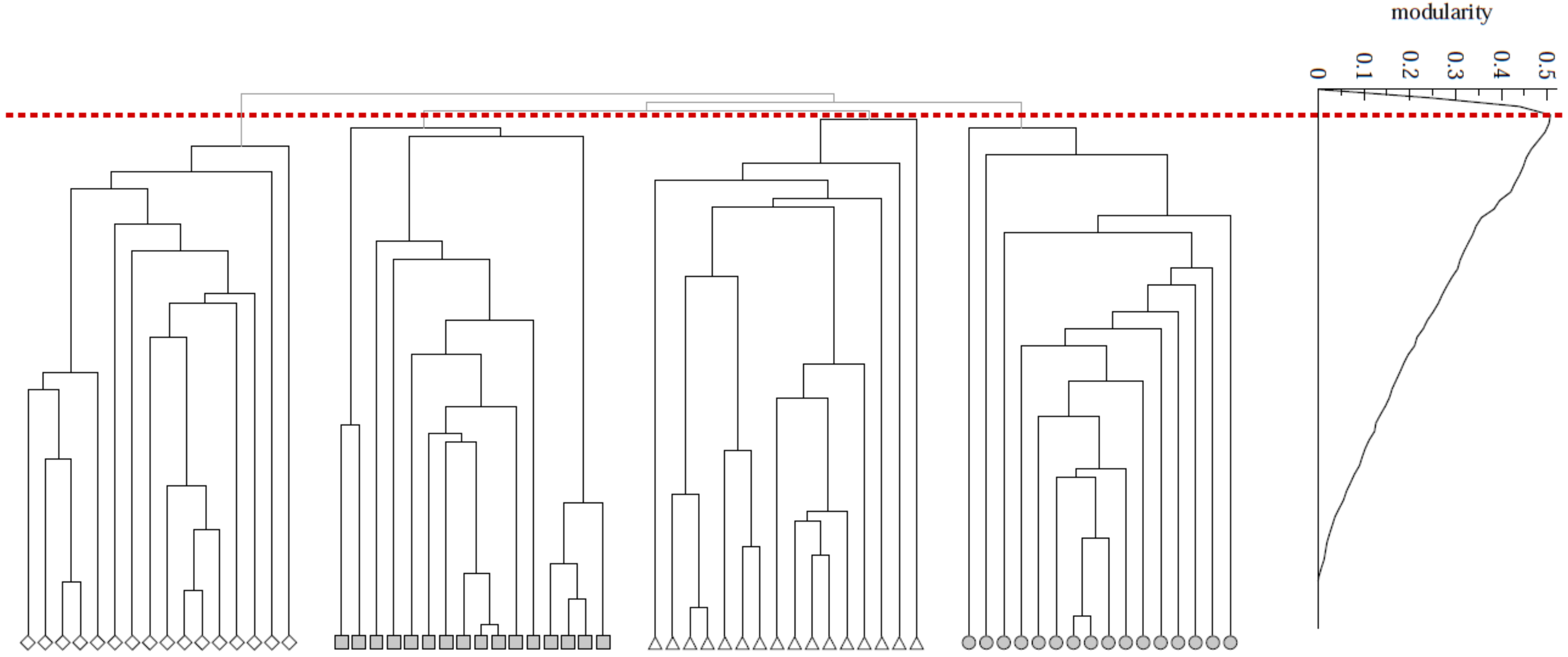
$$Q(G,S) = \underbrace{\frac{1}{2m}}_{\text{Normalizing: } -1 < Q < 1} \sum_{s \in S} \sum_{i \in s} \sum_{j \in s} \left(a_{ij} - \frac{k_i k_j}{2m}\right)$$

Modularity values take range $[-1, 1]$:

- Positive if the number of edges within groups exceeds the expected number
- $0.3 - 0.7 < Q$ means significant community structure

Modularity

→ Q is useful for selecting the number of clusters



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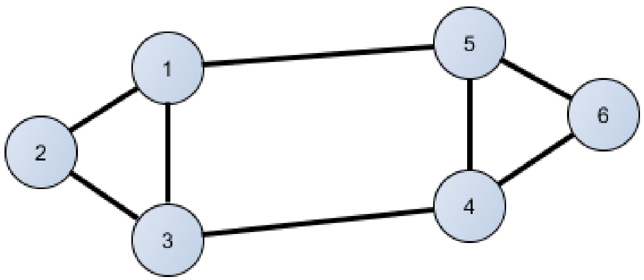
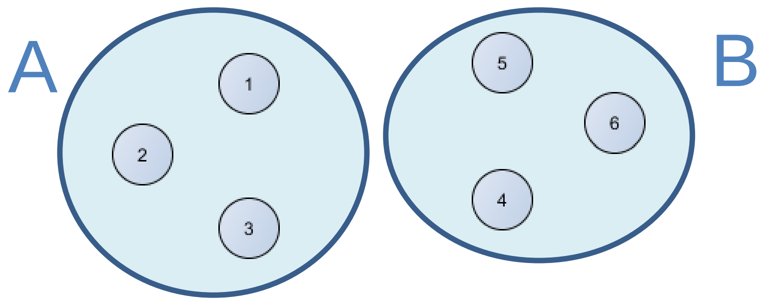
Partitioning of Graphs

Given an undirected Graph $G(V,E)$:



Bi-partitioning task:

- Divide vertices into two **disjoint** groups A, B



Questions:

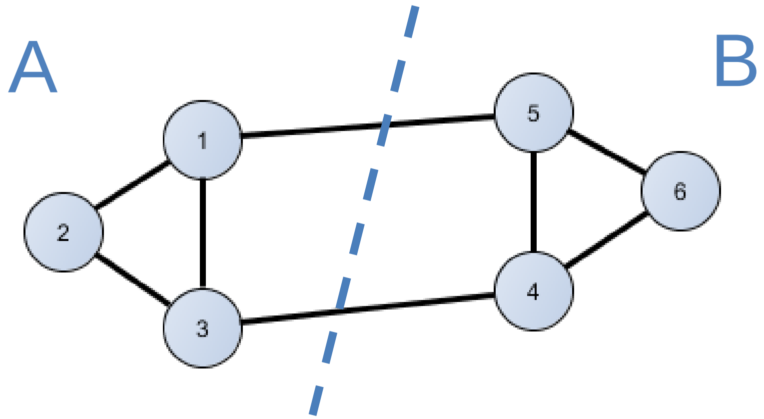
- How can we define 'good' partition of G?
- How can we efficiently identify such a partition?

Partitioning of Graphs

What makes a good partition?

- Maximize the number of within-group connections
- Minimize the number of between-group connections

Example:



Partitioning of Graphs

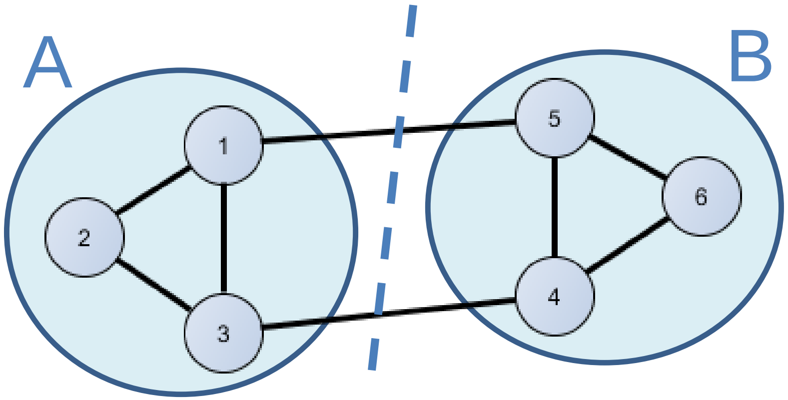
Graph Cuts

Express partitioning objectives as a function of the 'edge cut' of the partition

Cut: Set of edges with only one vertex in a group:

$$cut(A,B) = \sum_{i \in A, j \in B} w_{ij}$$

Example:



$\Rightarrow$ $\boldsymbol{cut(A,B) = 2}$

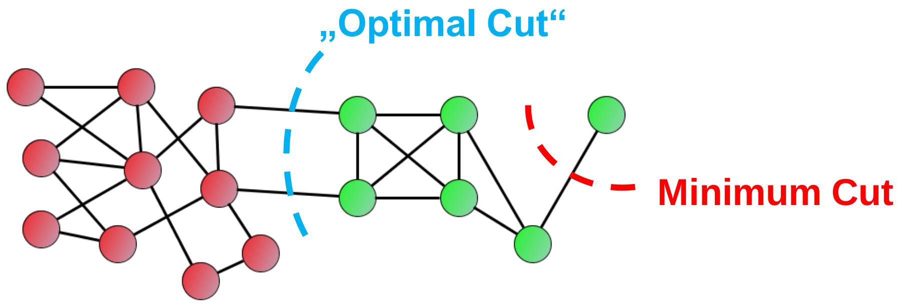
Partitioning of Graphs

Minimum-cut

Minimize weight of connections between groups:

$$\arg\min_{A,B} cut(A,B)$$

Example:



Problem:

- Only considers external cluster connections
- Does not consider internal cluster connectivity

Partitioning of Graphs - Graph Cuts

Normalized-cut: Connectivity between groups relative to the density of each group

$$\boldsymbol{ncut(A,B) = \frac{cut(A,B)}{vol(A)} + \frac{cut(A,B)}{vol(B)}}$$

$\boldsymbol{vol(X)}$**:** total weight of edges with at least one endpoint in X: $vol(X) = \sum_{i \in A} k_i$

→ Produces more balanced partitions

How to find a good partition efficiently?
Problem: Computing optimal cut is **NP-hard!**

Spectral Graph Partitioning

- **Adjacency matrix** of an undirected Graph G

  $a_{ij} = 1$ if (i, j) exist in G, else 0
- Vector $x \in R^n$ with components $(x_1, \dots, x_n)$

  Think of it as a label/value of each node of G

What is the meaning of $A * x$?

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} * \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad y_i = \sum_{j=1}^{n} a_{ij} * x_j = \sum_{(i,j) \in E} x_j$$

→ **Entry y_i is a sum of labels / values x_j of neighbors of i**

Spectral Graph Partitioning

What is the meaning of $\boldsymbol{A} * \boldsymbol{x}$?

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} * \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} * \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\boldsymbol{A} * \boldsymbol{x} = \boldsymbol{\lambda} * \boldsymbol{x} \quad \boldsymbol{Eigenvalue\ Problem}$$

Spectral Graph Theory:

- Analyze the ‚spectrum' of matrix representing G
- **Spectrum**: Eigenvectors x_i of a graph, ordered by the magnitude (strength) of their corresponding eigenvalues λ_i
- $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \leq \cdots \leq \lambda_n$

Spectral Graph Partitioning

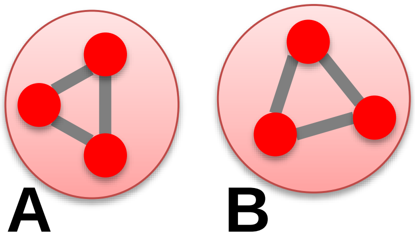
Intuition
Suppose all nodes in G have degree d and G is connected

What are some eigenvalues/vectors of G?
Eigenvalue Problem: $A * x = \lambda * x$ → find λ and x

- Let's try $x = (1, \dots, 1)$
- Then $A * x = (d, \dots, d) = \lambda * x$ → $\lambda = d$

Remember:
$$y_i = \sum_{j=1}^{n} a_{ij} * x_j = \sum_{(i,j)\in E} x_j$$

Spectral Graph Partitioning



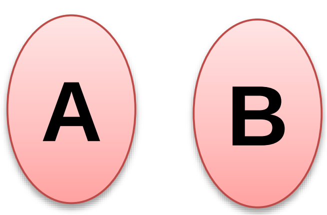
Intuition

What if G is not connected?

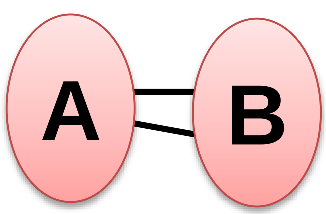
- G has 2 components, each d-regular

What are some eigenvectors?

- $x =$ put all 1s on A and 0s on B or vice versa
  - $x' = (1, \ldots, 1, 0, \ldots, 0)$, then $A * x' = (d, \ldots, d, 0, \ldots, 0)$
  - $x'' = (0, \ldots, 0, 1, \ldots, 1)$, then $A * x'' = (0, \ldots, 0, d, \ldots, d)$
  - → in both cases the corresponding $\lambda = d$



$\boldsymbol{\lambda_n = \lambda_{n-1}}$

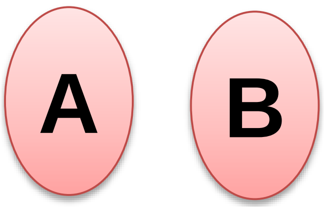


$\boldsymbol{\lambda_n - \lambda_{n-1} \approx 0}$

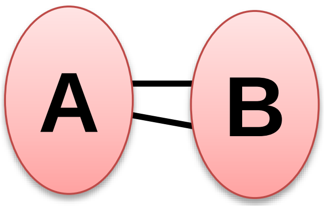
2nd largest eigval. λ_{n-1} now has value very close to λ_n

Spectral Graph Partitioning

Intuition



$\boldsymbol{\lambda_n = \lambda_{n-1}}$



$\boldsymbol{\lambda_n - \lambda_{n-1} \approx 0}$

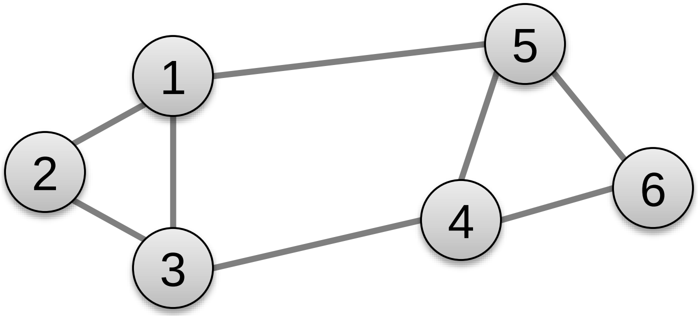
2nd largest eigval. λ_{n-1} now has value very close to λ_n

- If the graph is connected (right example) then we already know that $x_n = (1, \dots, 1)$ is an eigenvector
- Since eigenvectors are orthogonal then the components of x_{n-1} sum to 0
  - Why? → Because $\sum_i x_n[i] * \sum_i x_{n-1}[i] = 0$

- **General Idea**: we can look at the eigenvector of the 2nd largest eigenvalue and declare nodes with positive label in A and negative label in B

Spectral Graph Partitioning

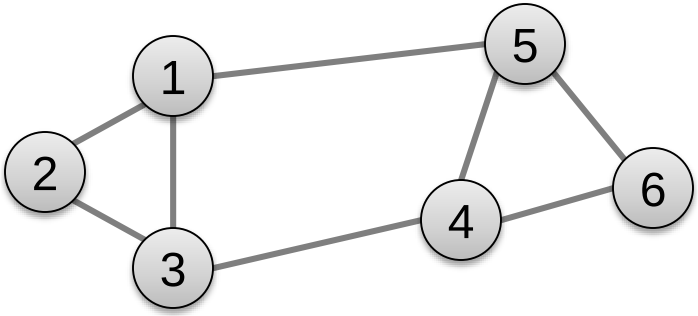
- **Adjacency matrix** A:
  - $\boldsymbol{n\ x\ n}$ matrix
  - $A = [a_{ij}], a_{ij} = 1$ if there is an edge between node i and j



	1	2	3	4	5	6
1	0	1	1	0	1	0
2	1	0	1	0	0	0
3	1	1	0	1	0	0
4	0	0	1	0	1	1
5	1	0	0	1	0	1
6	0	0	0	1	1	0

Spectral Graph Partitioning

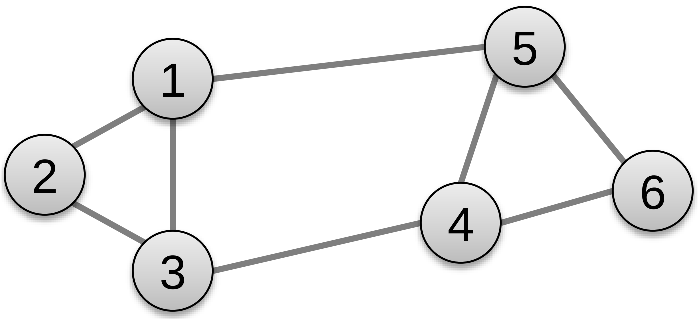
- **Degree matrix** D:
  - $\boldsymbol{n \; x \; n}$ diagonal matrix
  - $D = [d_{ii}], d_{ii} =$ degree of node i



	1	2	3	4	5	6
1	3	0	0	0	0	0
2	0	2	0	0	0	0
3	0	0	3	0	0	0
4	0	0	0	3	0	0
5	0	0	0	0	3	0
6	0	0	0	0	0	2

Spectral Graph Partitioning

- **Laplacian Matrix** L:
  - $\boldsymbol{n \ x \ n}$ symmetric matrix
  - $L = D - A$



	1	2	3	4	5	6
1	3	-1	-1	0	-1	0
2	-1	2	-1	0	0	0
3	-1	-1	3	-1	0	0
4	0	0	-1	3	-1	-1
5	-1	0	0	-1	3	-1
6	0	0	0	-1	-1	2

- Trivial eigenpair?
  - $X = (1, \dots, 1)$, then $L * x = 0$ and so $\lambda_1 = 0$

Spectral Clustering Algorithms

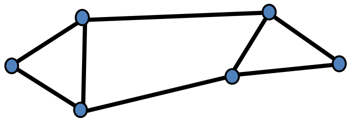
Three basic stages:

1. **Pre-processing**
   - Construct a matrix representation of the graph
2. **Decomposition**
   - Compute eigenvalues and eigenvectors of the matrix
   - Map each point point to a lower-dimensional representation on one or more eigenvectors
3. **Grouping**
   - Assign points to two or more clusters, based on the new representation

Spectral Clustering Algorithms

1. Pre-processing:

- Build Laplacian matrix L of the graph



	1	2	3	4	5	6
1	3	-1	-1	0	-1	0
2	-1	2	-1	0	0	0
3	-1	-1	3	-1	0	0
4	0	0	-1	3	-1	-1
5	-1	0	0	-1	3	-1
6	0	0	0	-1	-1	2

2. Decomposition:

- Find eigenvalues λ and eigenvectors x of the matrix L

λ=

0.0
1.0
3.0
3.0
4.0
5.0

X =

0.4	0.3	-0.5	-0.2	-0.4	-0.5
0.4	0.6	0.4	-0.4	0.4	0.0
0.4	0.3	0.1	0.6	-0.4	0.5
0.4	-0.3	0.1	0.6	0.4	-0.5
0.4	-0.3	-0.5	-0.2	0.4	0.5
0.4	-0.6	0.4	-0.4	-0.4	0.0

- Map vertices to lower-dimensional representation

1	0.3
2	0.6
3	0.3
4	-0.3
5	-0.3
6	-0.6

How do we now find the clusters?

Spectral Clustering Algorithms

3. Grouping:

- Sort components of reduced 1-dimensional vector
- Identify clusters by splitting the sorted vector in two (threshold ε)
- By choosing m vectors, there are max. 2^m clusters

→ How to choose a splitting point, i.e threshold ε?

- Naive approaches:
  - Split at $\varepsilon = 0$ or median value
- More expensive approaches:
  - Attempt to minimize normalized cut in 1-dimension
  - (sweep over ordering of nodes induces by the eigenvector)

1	0.3
2	0.6
3	0.3
4	-0.3
5	-0.3
6	-0.6

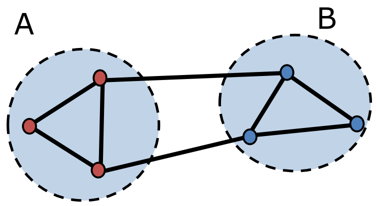
Split at $\varepsilon = 0$:

Cluster A: Positive points

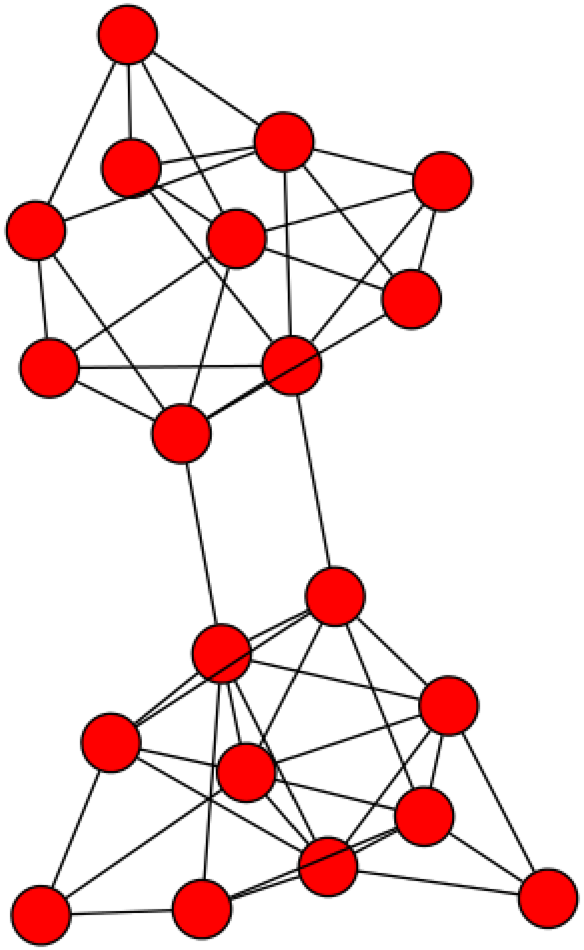
Cluster B: Negative points

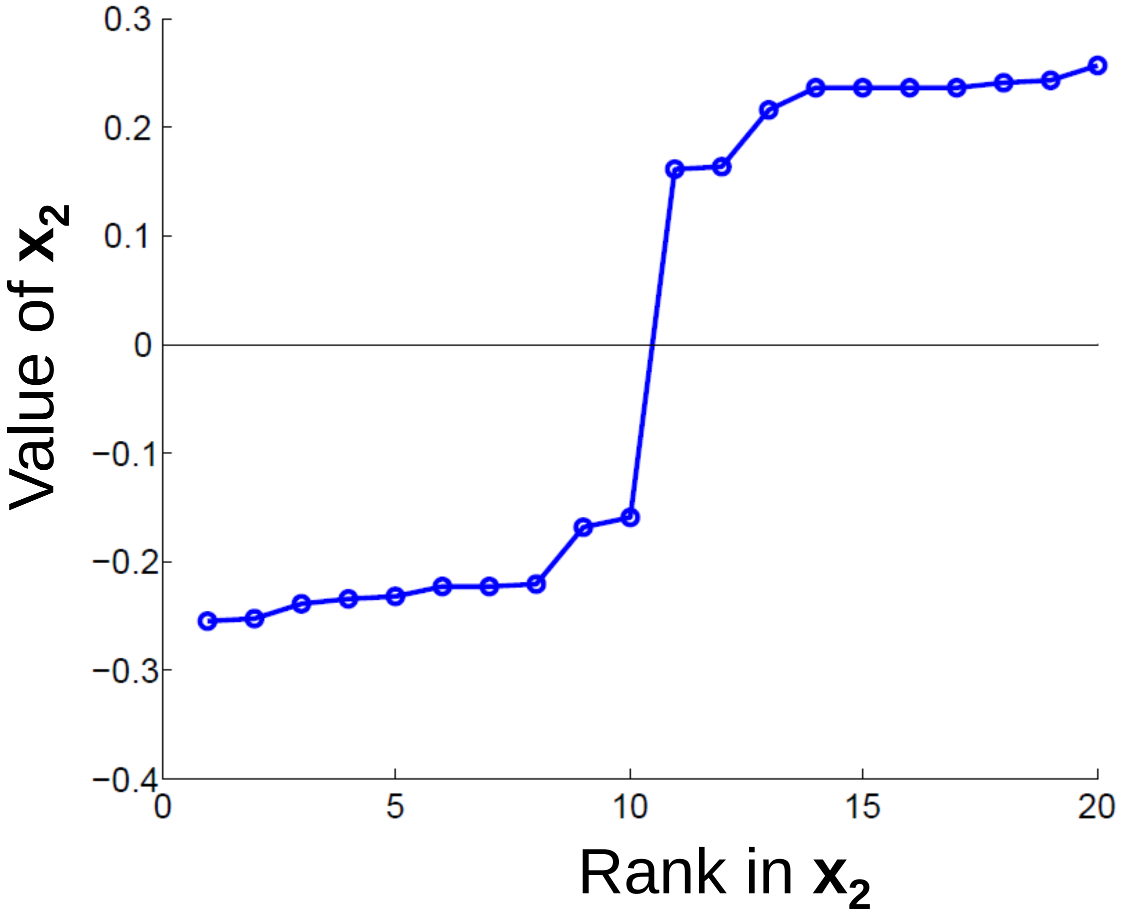
1	0.3
2	0.6
3	0.3

4	-0.3
5	-0.3
6	-0.6

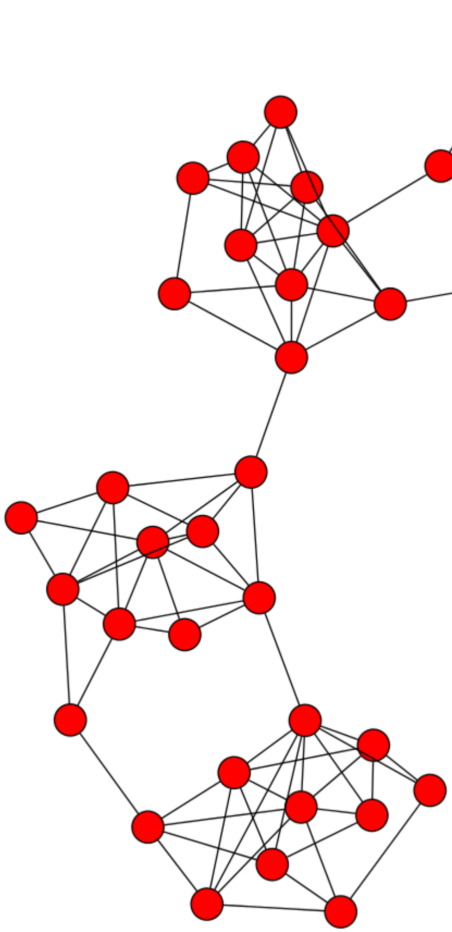


Spectral Clustering Algorithms

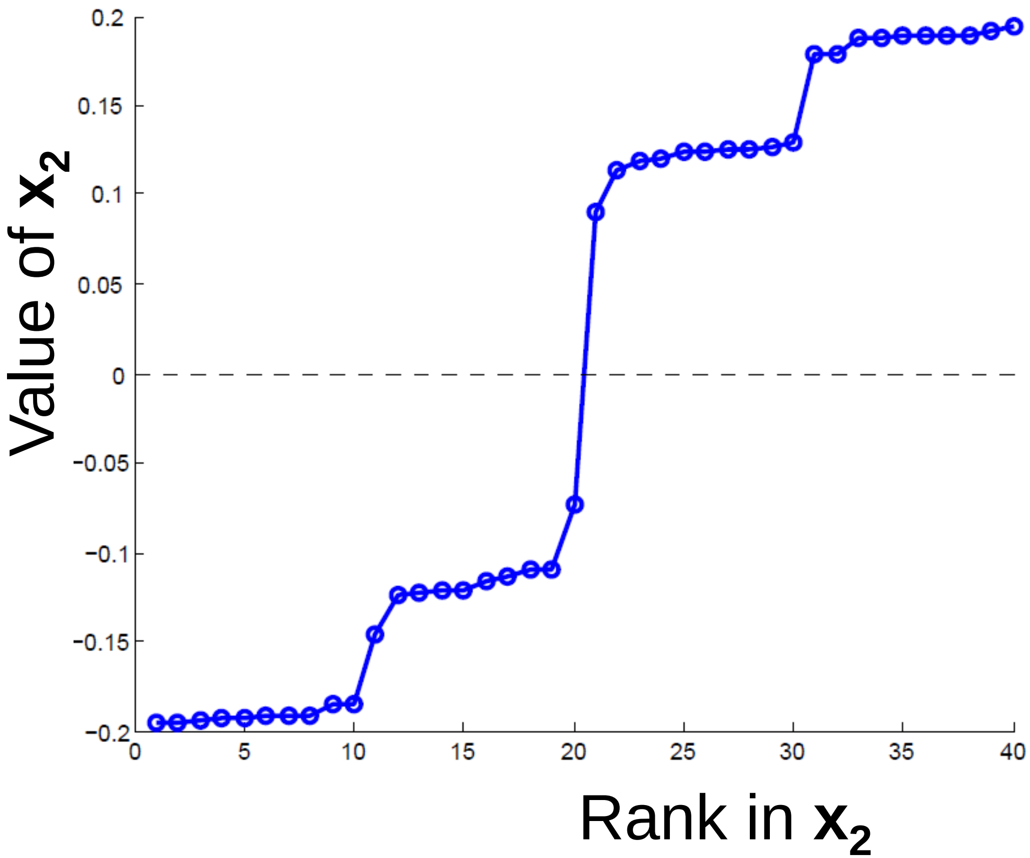




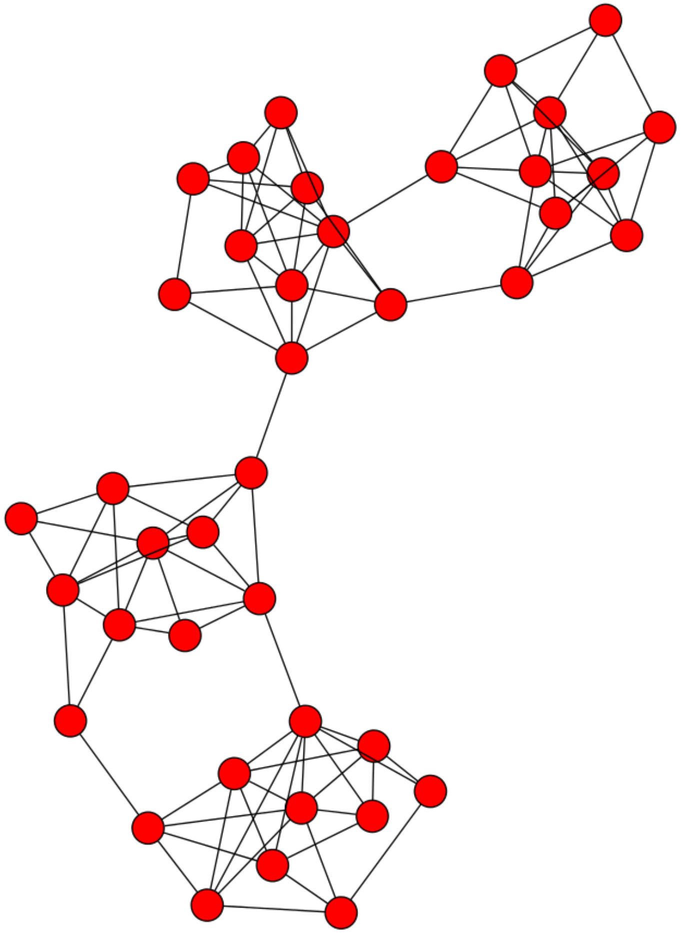
Spectral Clustering Algorithms

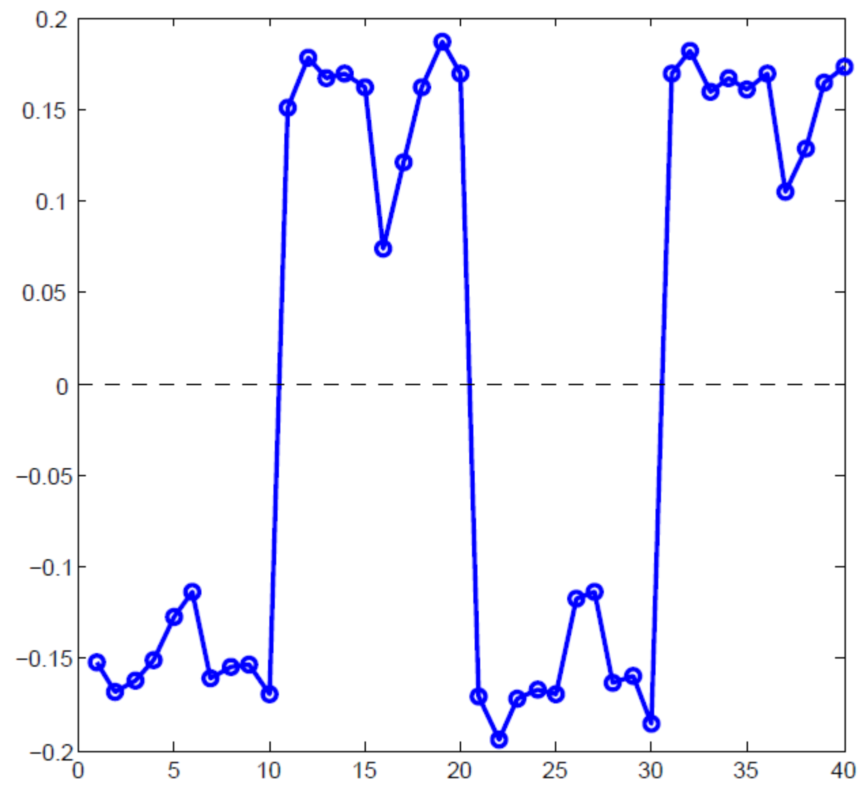


Components of x_2



Spectral Clustering Algorithms





Components of $\mathbf{x_3}$

Outline

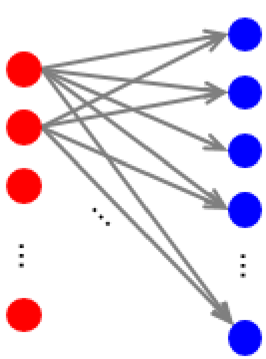
Community Detection

- Social networks
- Betweenness
  - Girvan-Newman Algorithm
- Modularity
- Graph Partitioning
  - Spectral Graph Partitioning
- Trawling

Trawling

- Goal: find small communities in huge graphs
- E.g. how to describe community/discussion in a Web

Example:

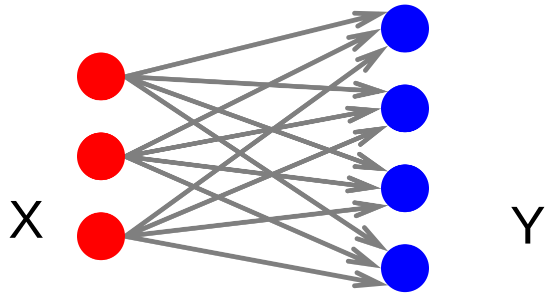


E.g. people talking about the same things or visited web pages

Problem definition:

Enumerate complete bipartite subgraphs $K_{s,t}$:

- All vertices in $K_{s,t}$ can be partitioned in two sets. Each vertex in the first set of size ***s*** is linked to each vertex in second set of size ***t***
- Where $K_{s,t}$: *s* nodes on the "left" where each links to the same *t* other nodes on the "right"

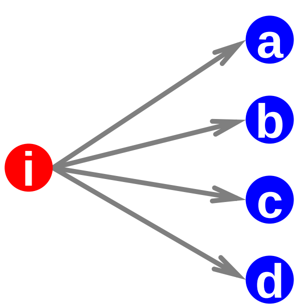


|X| = s = 3
|Y| = t = 4

Frequent Itemset Analysis – Market Basket Analysis

- ***Market***: Universe U of n items
- ***Baskets***: subsets of U: $S_1, S_2, \ldots, S_m \subseteq U$
  - (S_i is a set of items one person bought)
- ***Support:*** frequency threshold
- ***Goal:*** **Find all subsets T s.t. $T \subseteq S_i$ of at least f sets S_i**
  - (items in T were bought together at least f times)

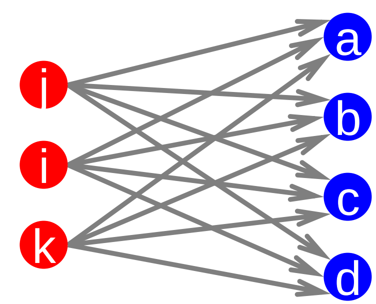
Frequent itemsets = complete bipartite graphs



S_i={a,b,c,d}

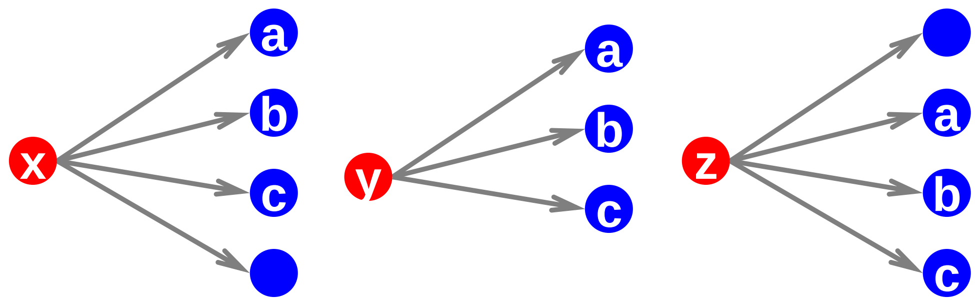
View each node i as a set S_i of nodes i points to

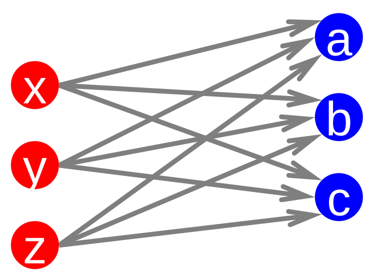
$K_{s,t}$ = a set Y of size t that occurs in s sets S_i



Analysis of Large Graphs - Trawling

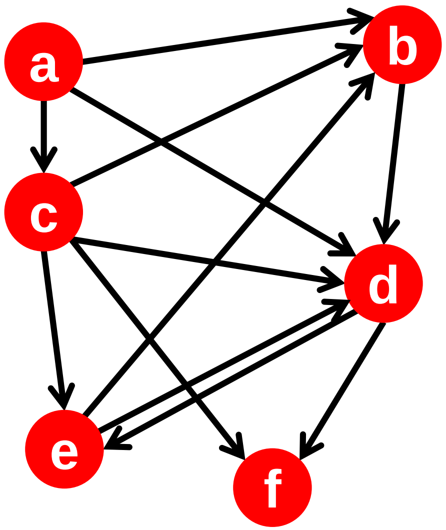
E.g. Bipartite subgraph $K_{3,4}$ a **frequent itemset *Y={a,b,c}*** of supp ***s*.** So, there are ***s*** nodes that link to all of **{a,b,c}**:





We found $K_{s,t}$!
$K_{s,t}$ = a set ***Y*** of size ***t***
that occurs in *s* sets S_i

Analysis of Large Graphs - Trawling



Itemsets:

a = {b,c,d}
b = {d}
c = {b,d,e,f}
d = {e,f}
e = {b,d}
f = {}

Frequent itemsets support > 1

{b,d}: support 3
{e,f}: support 2